An Introduction to
ANALYTIC COMBINATORICS

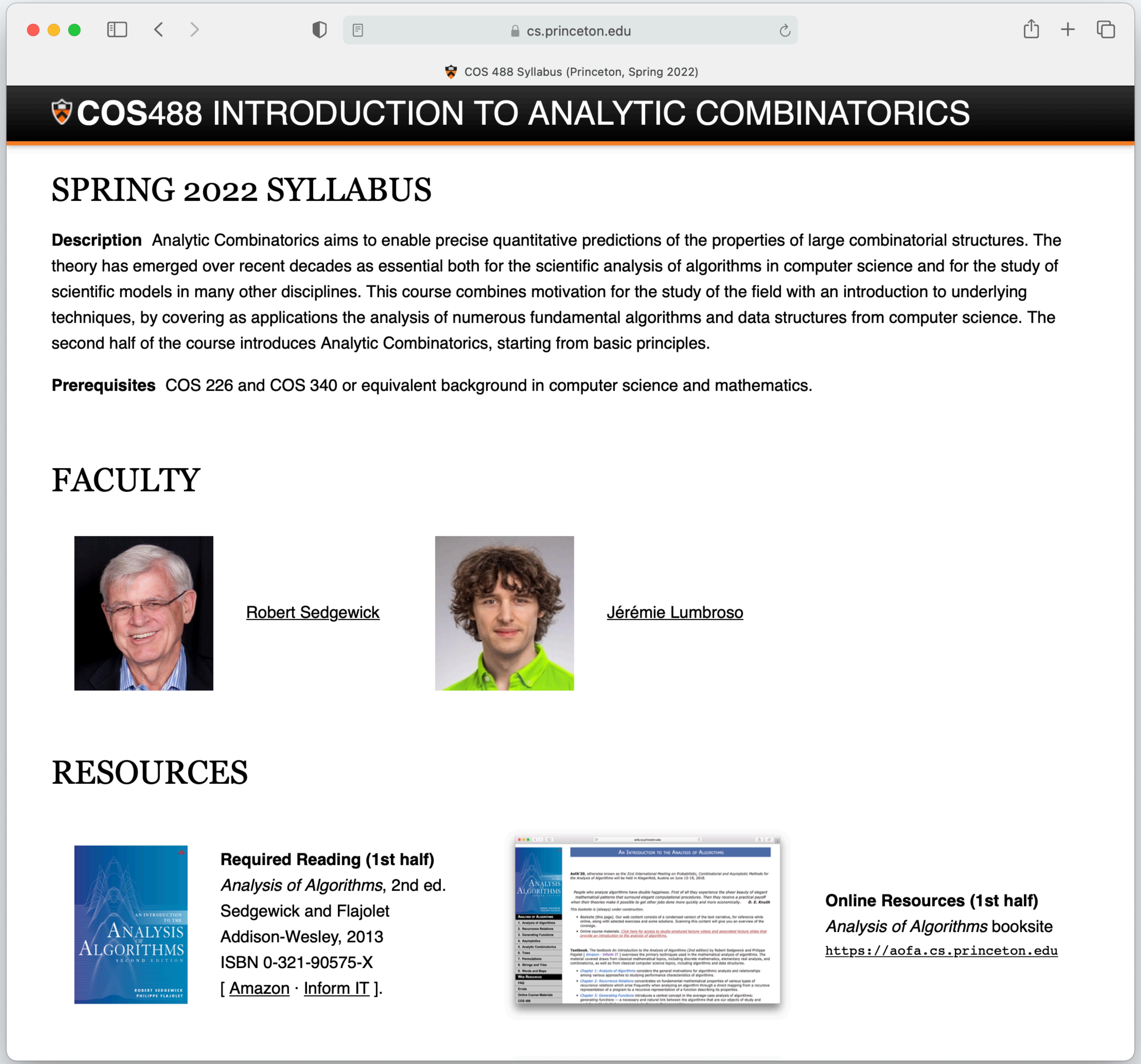
Computer Science 488

Robert Sedgewick
Jérémie Lumbroso

Course website: `https://www.cs.princeton.edu/courses/archive/spring22/cos488`

Our "contract" with you

- Syllabus.
- Schedule.
- Policies.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COS 488 Syllabus (Princeton, Spring 2022)

COS488 INTRODUCTION TO ANALYTIC COMBINATORICS

SPRING 2022 SYLLABUS

Description Analytic Combinatorics aims to enable precise quantitative predictions of the properties of large combinatorial structures. The theory has emerged over recent decades as essential both for the scientific analysis of algorithms in computer science and for the study of scientific models in many other disciplines. This course combines motivation for the study of the field with an introduction to underlying techniques, by covering as applications the analysis of numerous fundamental algorithms and data structures from computer science. The second half of the course introduces Analytic Combinatorics, starting from basic principles.

Prerequisites COS 226 and COS 340 or equivalent background in computer science and mathematics.

FACULTY

Robert Sedgewick

Jérémie Lumbroso

RESOURCES

Required Reading (1st half)
Analysis of Algorithms, 2nd ed.
Sedgewick and Flajolet
Addison-Wesley, 2013
ISBN 0-321-90575-X
[Amazon · Inform IT].

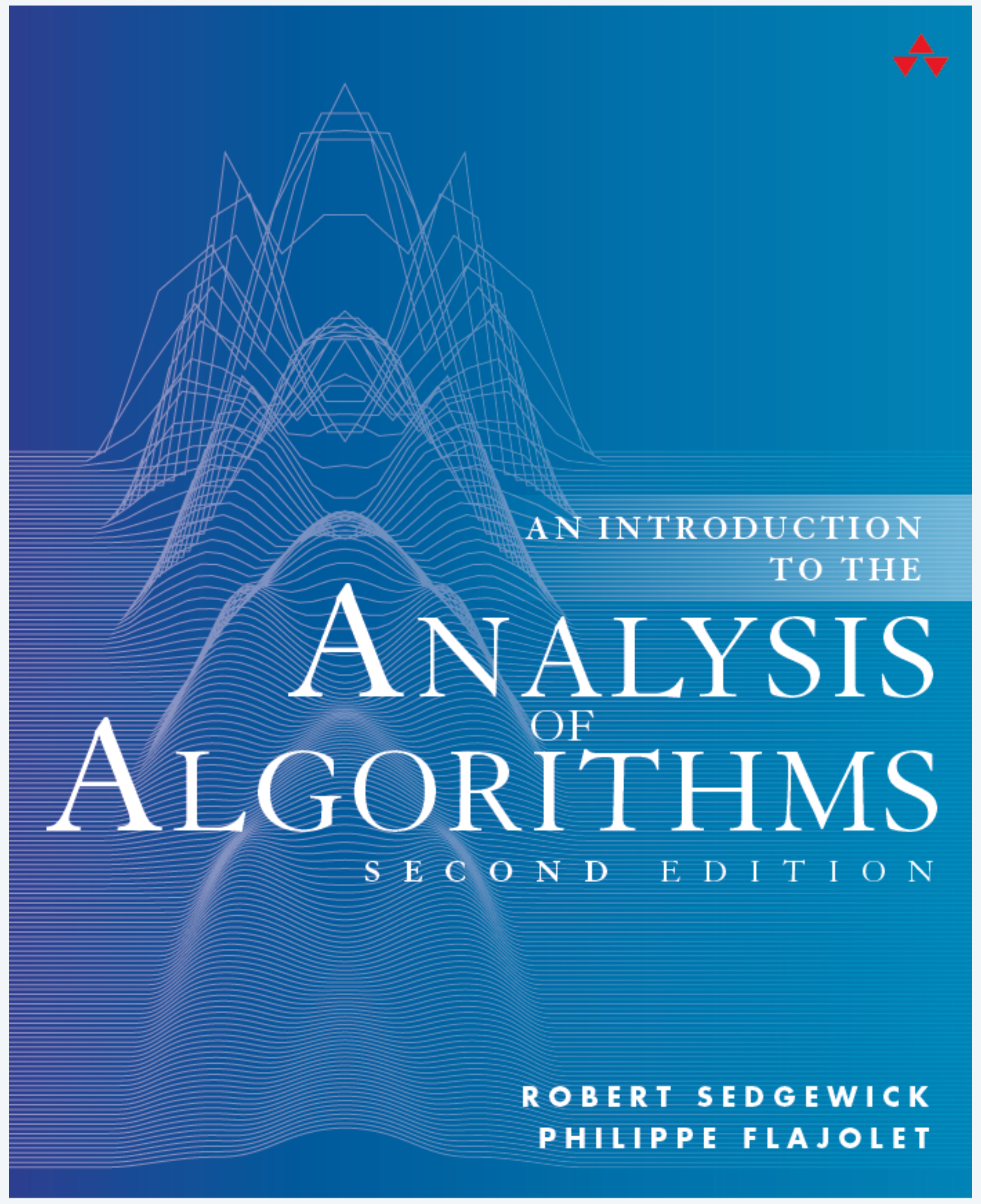
Online Resources (1st half)
Analysis of Algorithms booksite
`https://aofa.cs.princeton.edu`

Textbooks

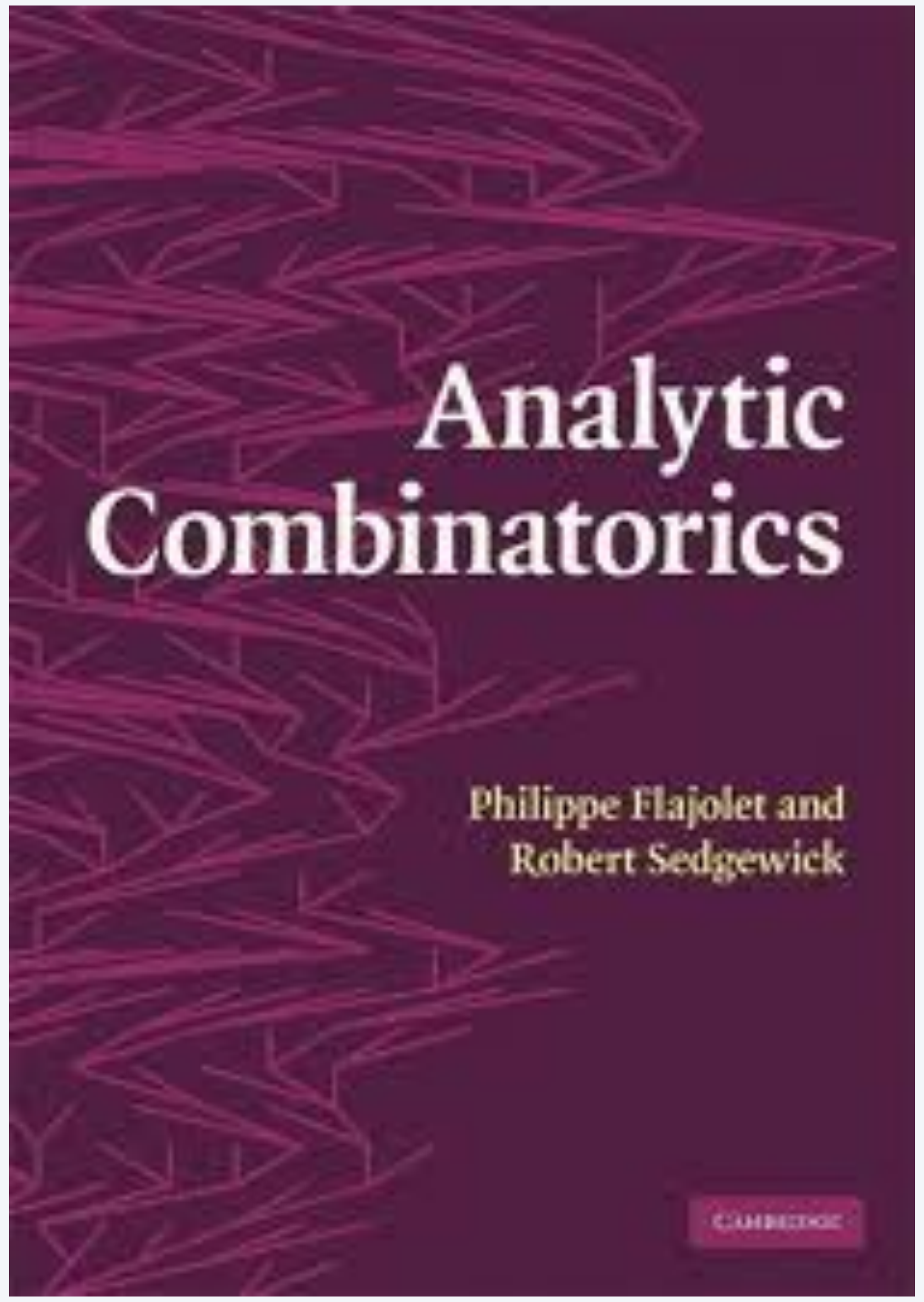
Flajolet-Sedgewick textbooks are the basis of this course.

- Full coverage of course material.
- You are *not* expected to read and understand every word.
- Advice: skim to get an idea of what's there, read what interests you.

First half



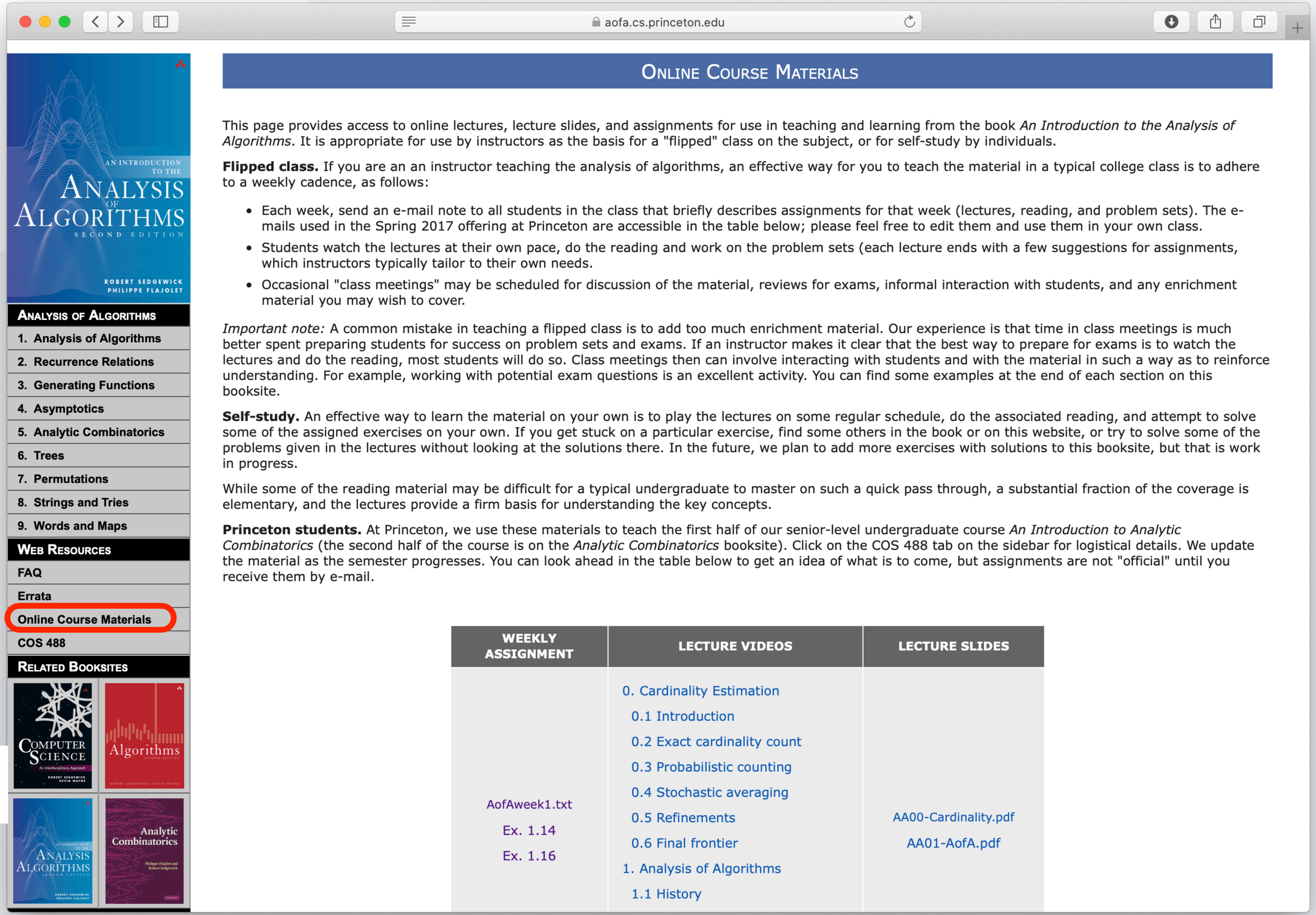
Second half



Booksites

The course is driven by online resources.

- Summary of content.
- Weekly problem sets.
- Lecture videos.
- Lecture slides.



http://aofa.cs.princeton.edu

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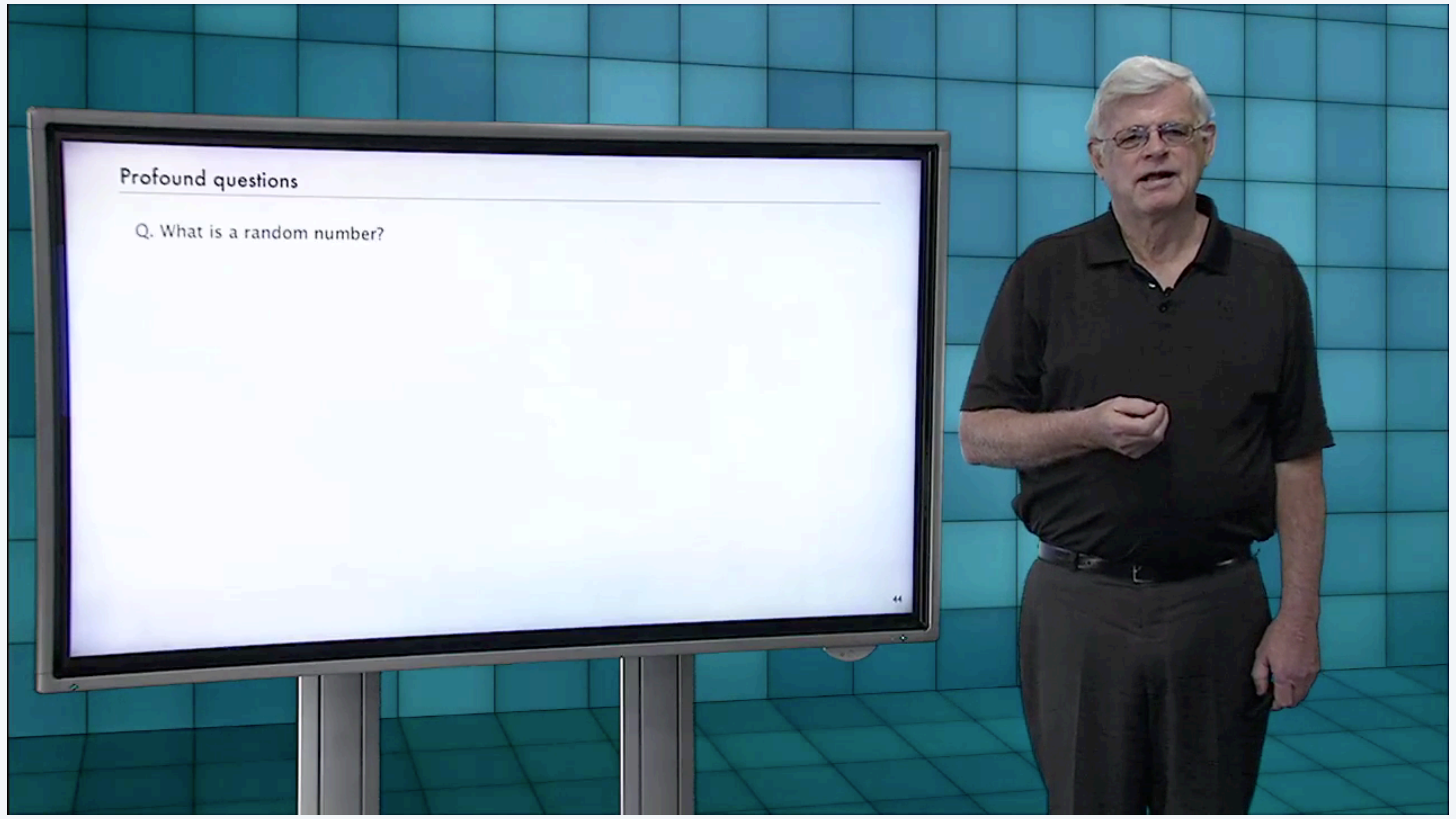
bookmark this page

Video lectures

Online lecture materials are on the booksite

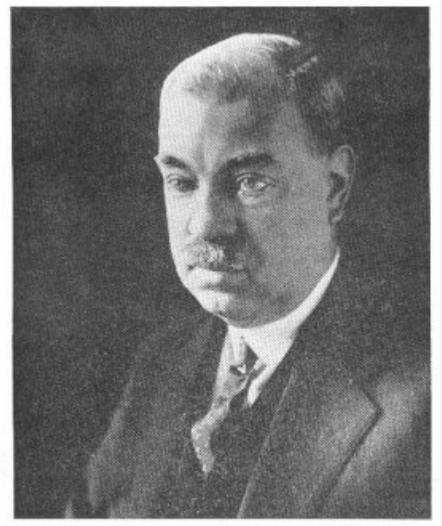
- Copies of slides (.pdf and 4-up).
- Studio-produced videos.
- ***No live lectures.*** ← Even before the pandemic
- Exams are on lecture content.



Why? Because you can

- Fit lectures into your schedule more easily.
- Slow down or rewind if you get lost.
- Speed up or skip if you get bored.
- Focus on understanding, not note-taking.

" Lecturing is that mysterious process by means of which the contents of the note-book of the professor are transferred ... to the note-book of the student without passing through the mind of either. "



– Edwin Emery Slosson

Bottom line. Online format has *proven* to be more effective and efficient than live lectures.

Assessments

Your grade is based on these components.

- Eleven problem sets, due Thursdays
- Questions and answers, due with problem sets.
- Two "review" problem sets.

Problem sets.

- Four problems (typical).
- Discuss approaches with others.
- Prepare submitted solutions without help.

"Review" problem sets

- One question per chapter.
- Assess understanding of basic concepts.
- Work alone.

due dates

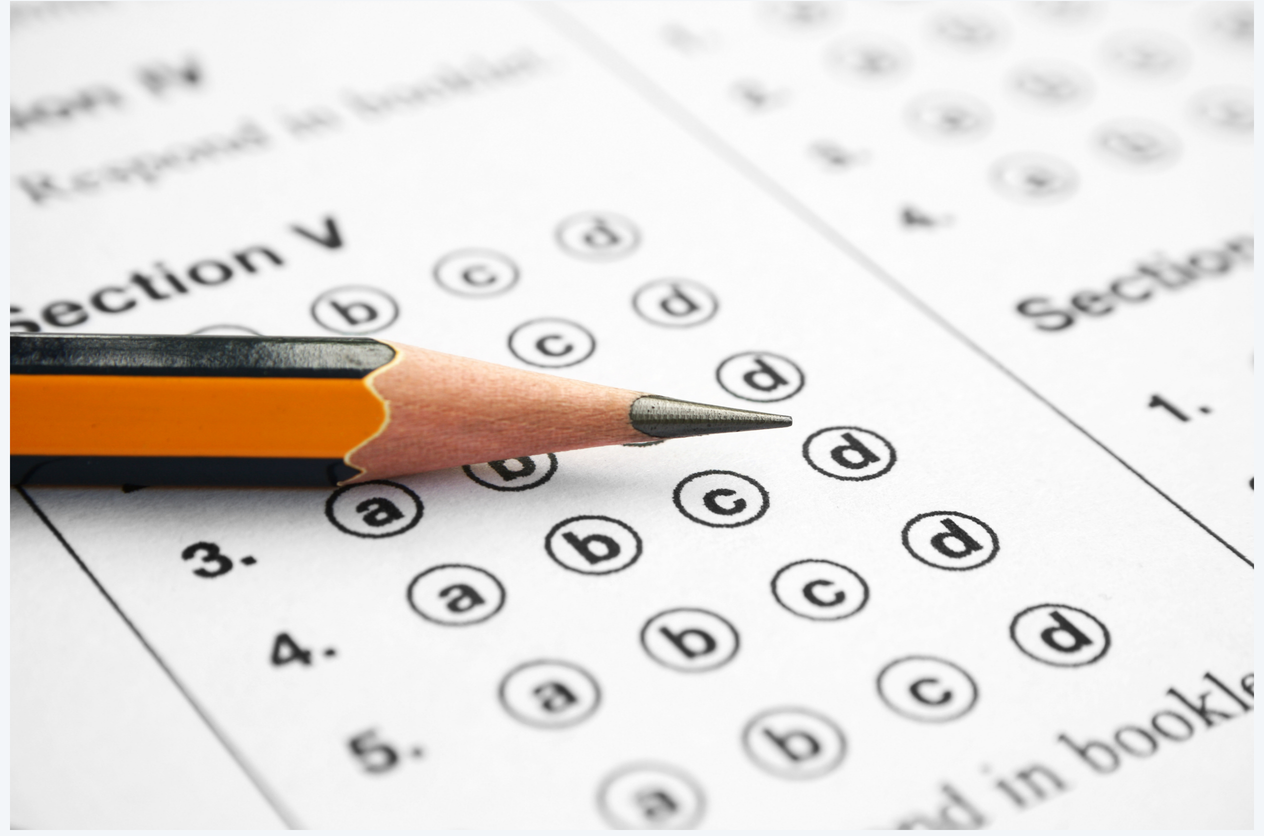
Jan 27	AofA week 1
Feb 3	AofA week 2
Feb 10	AofA week 3
Feb 17	AofA week 4
Feb 24	AofA week 5
Mar 3	AofA week 6
Mar 17	AofA Review
Mar 24	AC week 1
Mar 31	AC week 2
Apr 7	AC week 3
Apr 14	AC week 4
Apr 21	AC week 5
May 3	AC Review

“Questions and Answers” (Q&As)

One goal of COS 488 will be to learn develop good questions for assessment.

Properties of a good assessment question.

- Easy to understand.
- Easy to grade.
- Not trivial.
- Solvable in a reasonable amount of time.
- Test understanding of an important topic.
- “Fair” (no tricks)
- Teach something (optional but desirable)



Developing questions with answers *cements understanding* and is a skill everyone should learn.

Types of questions

Problem set.
- Similar to worked examples in lecture.
- Goal is to build confidence in understanding.
- Solvable in a hour or less.
- Partial credit with feedback.

Review.
- Straightforward application of lecture material.
- Goal is to assess understanding of basics.
- Solvable in 15–30 minutes.
- Short answer, easy to grade.
- Little or no partial credit.

Challenge (PF "Notes" in AC).
- "Surprise" facts related to content.
- Approach to solution not obvious.
- Might take a long time to solve.

Inclass exam.
- Easy to understand.
- Easy to grade.
- Solvable in 10 minutes or less.

no inclass exams in COS 488 anymore

Online.
- All of the above.
- Some olutions for self-assessment.

Q. Solve the following recurrence

$$F_N = N^2 + \frac{1}{N} \sum_{1 \le k \le N} (F_{k-1} + F_{N-k}) \quad \text{with} \quad F_0 = 0.$$

Not suitable for an inclass exam [too much detail in the calculations].

Maybe OK for review question.

Simplifying the recurrence

$$C_N = N + 1 + \sum_{1 \le k \le N} \frac{1}{N}(C_{k-1} + C_{N-k})$$

both sums are $C_0 + C_1 + \ldots + C_{N-1}$

Apply symmetry.

$$C_N = N + 1 + \frac{2}{N} \sum_{1 \le k \le N} C_{k-1}$$

Multiply both sides by N.

$$NC_N = N(N+1) + 2 \sum_{1 \le k \le N} C_{k-1}$$

Subtract same formula for *N*–1.

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

Collect terms.

$$NC_N = (N+1)C_{N-1} + 2N$$

$$Nt_N - (N-1)t_{N-1}$$

AofA Intro Q&A 1 (better version, "easy" review question)

Q. Match each “toll function” at left with the order of growth of the solution at right for the Quicksort recurrence

$$F_N = t_N + \frac{1}{N}\sum_{1\le k\le N}\left(F_{k-1} + F_{N-k}\right) \quad \text{with} \quad F_0 = 0$$

$$NC_N - (N+1)C_{N-1} = 3N^2 - 3N + 1$$

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + 3 + \ldots$$

t_N
1
N
0
N^2
$2N + 1$

order of growth of F_N
0
constant
linear
linearithmic
quadratic
cubic

A good question *must* avoid answers that depend on detailed calculations.

AofA Recurrences Q&A 1

Q. Solve the recurrence

$$na_n = (n-3)a_{n-1} + n \quad \text{for} \quad n \geq 3 \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \leq 2$$

n	a_n
3	1
4	5/4
5	3/2
6	7/4

summation factor

A.

$$n\overline{(n-1)(n-2)}a_n = \overline{(n-1)(n-2)}(n-3)a_{n-1} + n\overline{(n-1)(n-2)} \quad \text{for} \quad n \geq 3$$

$$\binom{n}{3}a_n = \binom{n-1}{3}a_{n-1} + \binom{n}{3} \quad \text{for} \quad n \geq 3$$

$$= \sum_{3 \leq k \leq n} \binom{k}{3} \quad = \binom{n+1}{4}$$

$$a_n = \frac{n+1}{4}$$

Suitable for an inclass exam, or Part 1 of a review quiestion.

better: $na_n = (n-3)a_{n-1} + 4n$

AofA Recurrences Q&A 1 (review question version)

Q. Solve the recurrence

$$na_n = (n-3)a_{n-1} + n \quad \text{for} \quad n \geq \mathbf{4} \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \leq \mathbf{3}$$

n	a_n
4	1
5	7/5
6	17/10
7	

A.

$$n(n-1)(n-2)a_n = (n-1)(n-2)(n-3)a_{n-1} + n(n-1)(n-2) \quad \text{for} \quad n \geq \mathbf{4}$$

$$\binom{n}{3}a_n = \binom{n-1}{3}a_{n-1} + \binom{n}{3} \quad \text{for} \quad n \geq \mathbf{4}$$

$$= \sum_{\mathbf{4} \leq k \leq n} \binom{k}{3} = \binom{n+1}{4} - 1$$

$$a_n = \frac{n+1}{4} - \frac{1}{\binom{n}{3}}$$

Tough to do and check in 8 minutes, but OK with time to check calculations carefully.

AofA Recurrences Q&A 1 (version used on MOOC)

Q. Suppose that

$$na_n = (n-3)a_{n-1} + n \quad \text{for} \quad n \geq 3 \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \leq 2$$

What is the value of a_{999} ?

A. 250

Might be viewed as too tricky for an inclass exam, but maybe OK for a review question.

AofA Recurrences Q&A 2 (inclass exam question)

Q. Which of the following is true of the number of compares used by Mergesort ?

$$C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N \quad \text{for} \quad N \geq 2 \quad \text{with} \quad C_1 = 0$$

Order of growth is $N \lg N$	**T**
Exactly $N \lg N$ when N is a power of 2	**T**
Is equal to the number of 1s in the binary representation of the numbers $< N$	**F**
Has periodic behavior	**T**
Is less than $N \lg N + N/4$ for all N	**T**

Some questions are of the form: *Did you watch the lectures and/or do the reading*?

AofA GFs Q&A 1 (easy)

Q. Match each of the following sequences with their OGF.

`0, 0, 1, 3, 6, 10, ...`

`0, 0, 1/2, 0, 1/4, 0, 1/6, ...`

`1, 3, 9, 27, 81, 243, ...`

`1, 1 + 1/2, 1 + 1/2 + 1/3, ...`

`3, 3, 3, 3, 3, ...`

$$\frac{1}{1-3z}$$

$$\frac{z^2}{(1-z)^3}$$

$$\ln\frac{1}{1-z^2}$$

$$\frac{3}{1-z}$$

$$\ln\frac{1}{1-2z}$$

$$\frac{1}{1-z}\ln\frac{1}{1-z}$$

$$\frac{1}{(1-z)^3}$$

Q. Suppose that a_n satisfies $a_n = 9a_{n-1} - 20a_{n-2}$ for $n > 1$ with $a_0 = 0$ and $a_1 = 1$

What is $\lim\limits_{n\to\infty} a_n/a_{n+1}$?

A. **5**

$$a(z) = \frac{z}{1 - 9z + 20z^2} = \frac{z}{(1-4z)(1-5z)} = \frac{1}{1-5z} - \frac{1}{1-4z}$$

$$a_n = 5^n - 4^n$$

Q. Fill the circle corresponding to the value of

$$[z^n]\sum_{0\le k\le n}\binom{2k}{k}\binom{2n-2k}{n-k}$$

and justify your answer.

○ 2^n ● 4^n ○ $2^{n/2}$

It is $[z^n]\Big(\dfrac{1}{\sqrt{1-4z}}\Big)^2$

"Questions and Answers" (Q&As)

You must submit a ***question*** *(and answer) on the week's content with each week's assignment.*

Properties of a good assessment question.

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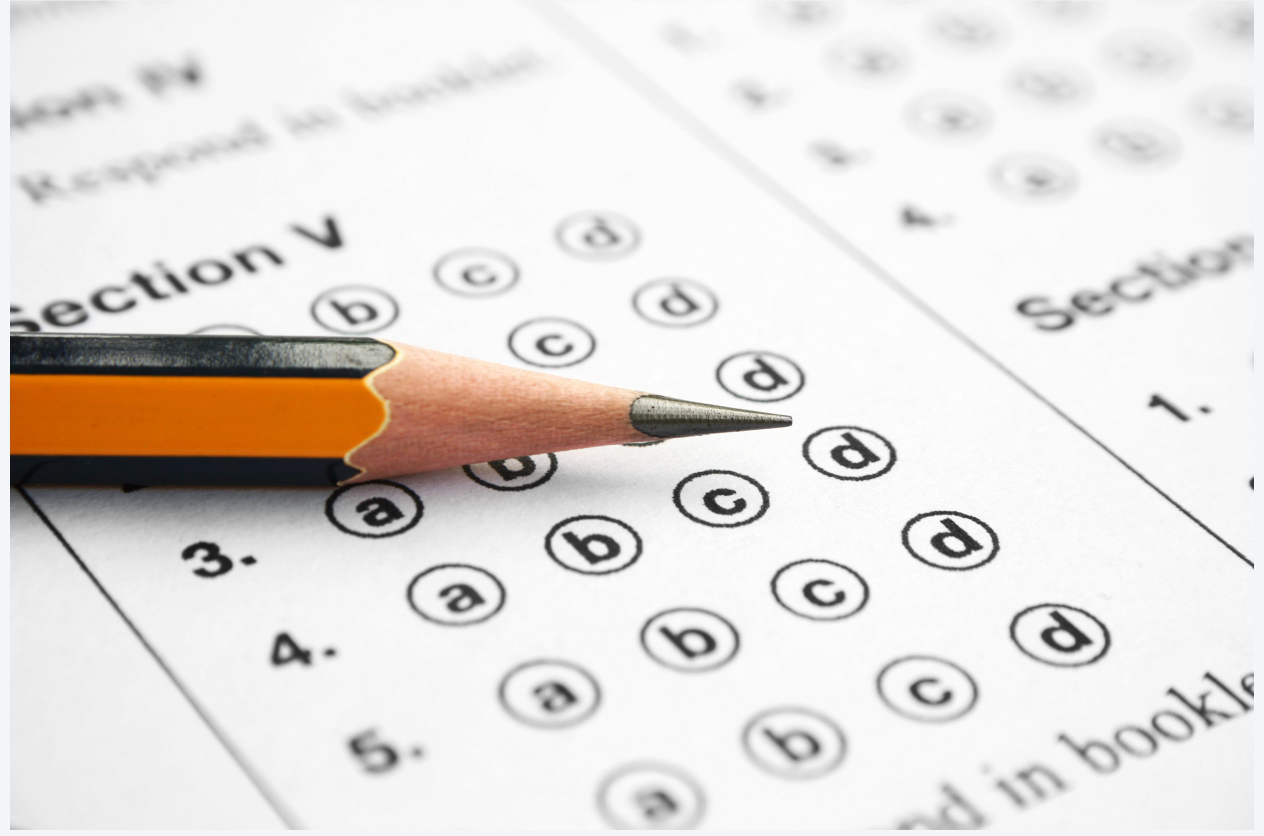
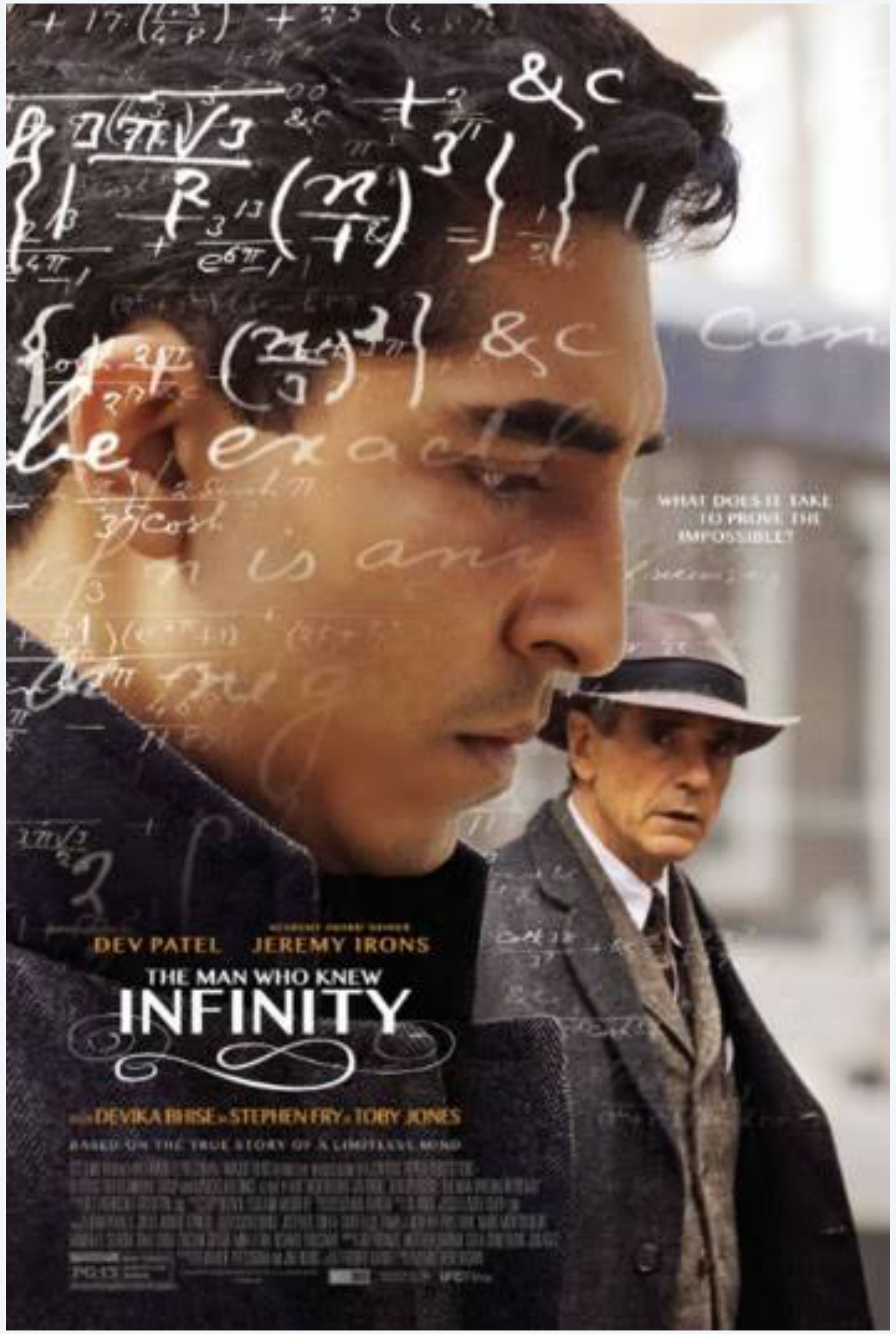


Developing questions with answers *cements understanding* and is a skill everyone should learn.

10% of course grade.

Enrichment I

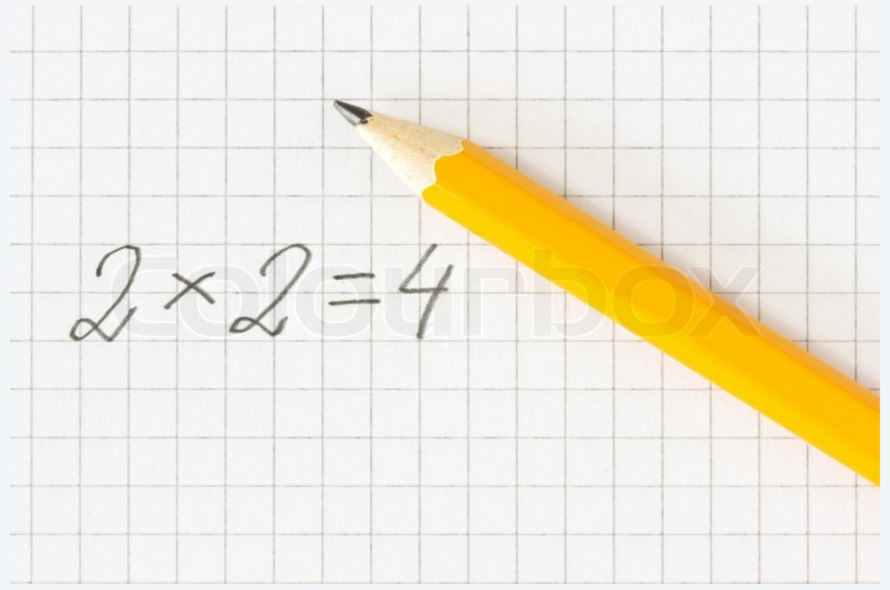


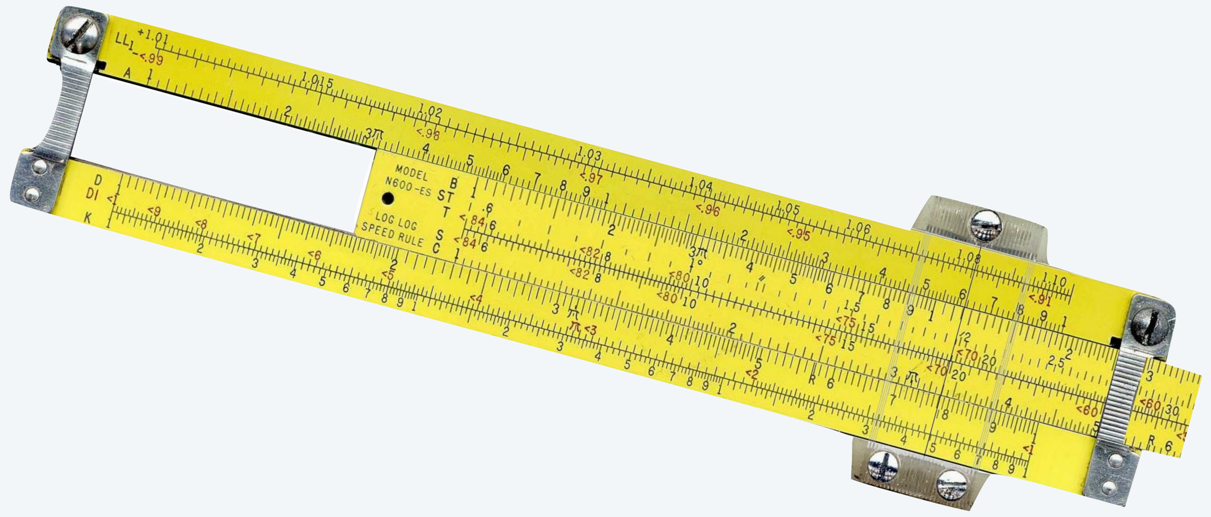
"An equation means nothing to me unless it expresses a thought of God."

– *Srinivasa Ramanujan*

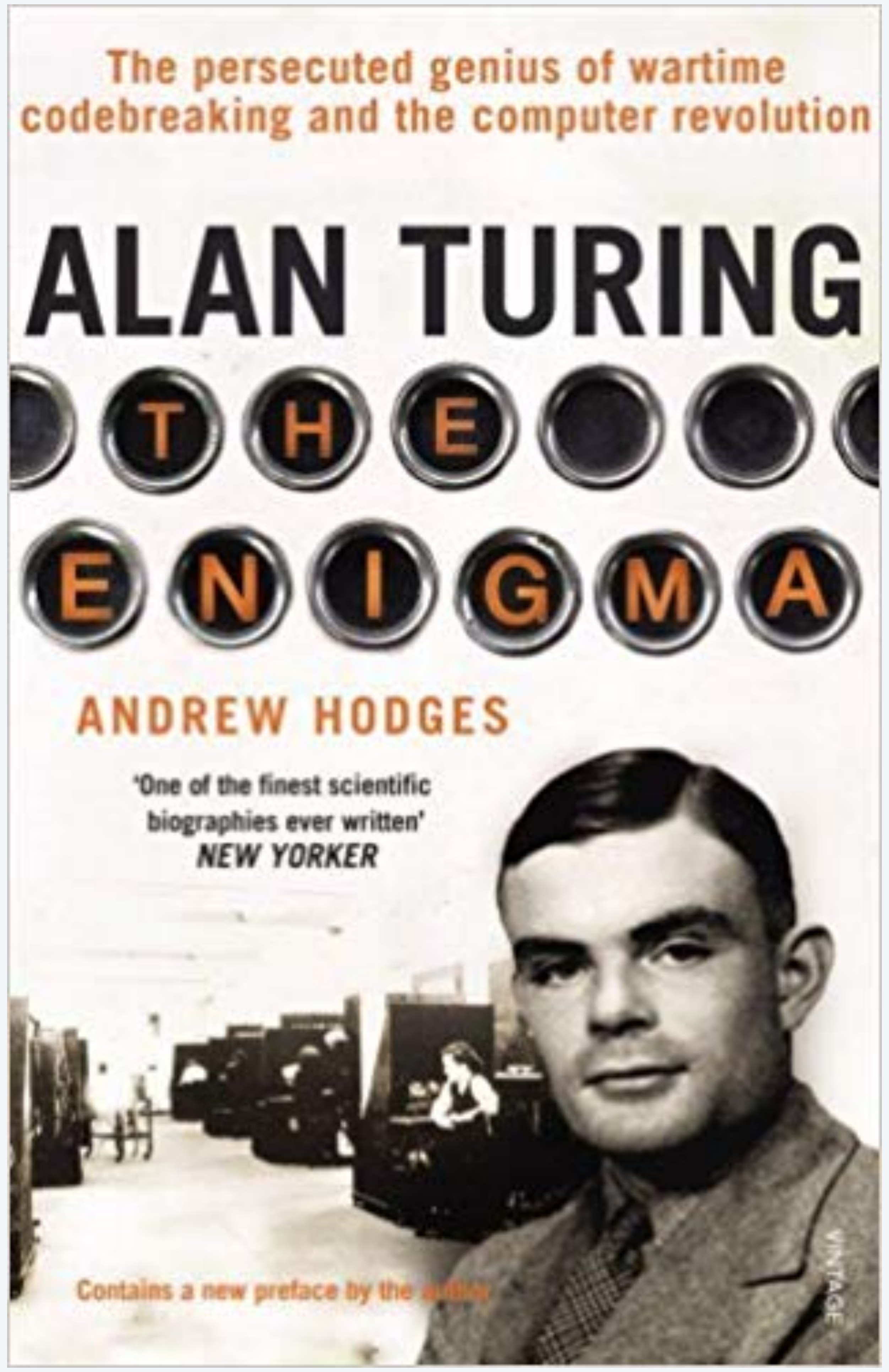
$$Q(N) \equiv \sum_{1 \le k \le N} \frac{N!}{(N-k)!N^k} = \sqrt{\frac{\pi N}{2}} + O(1)$$

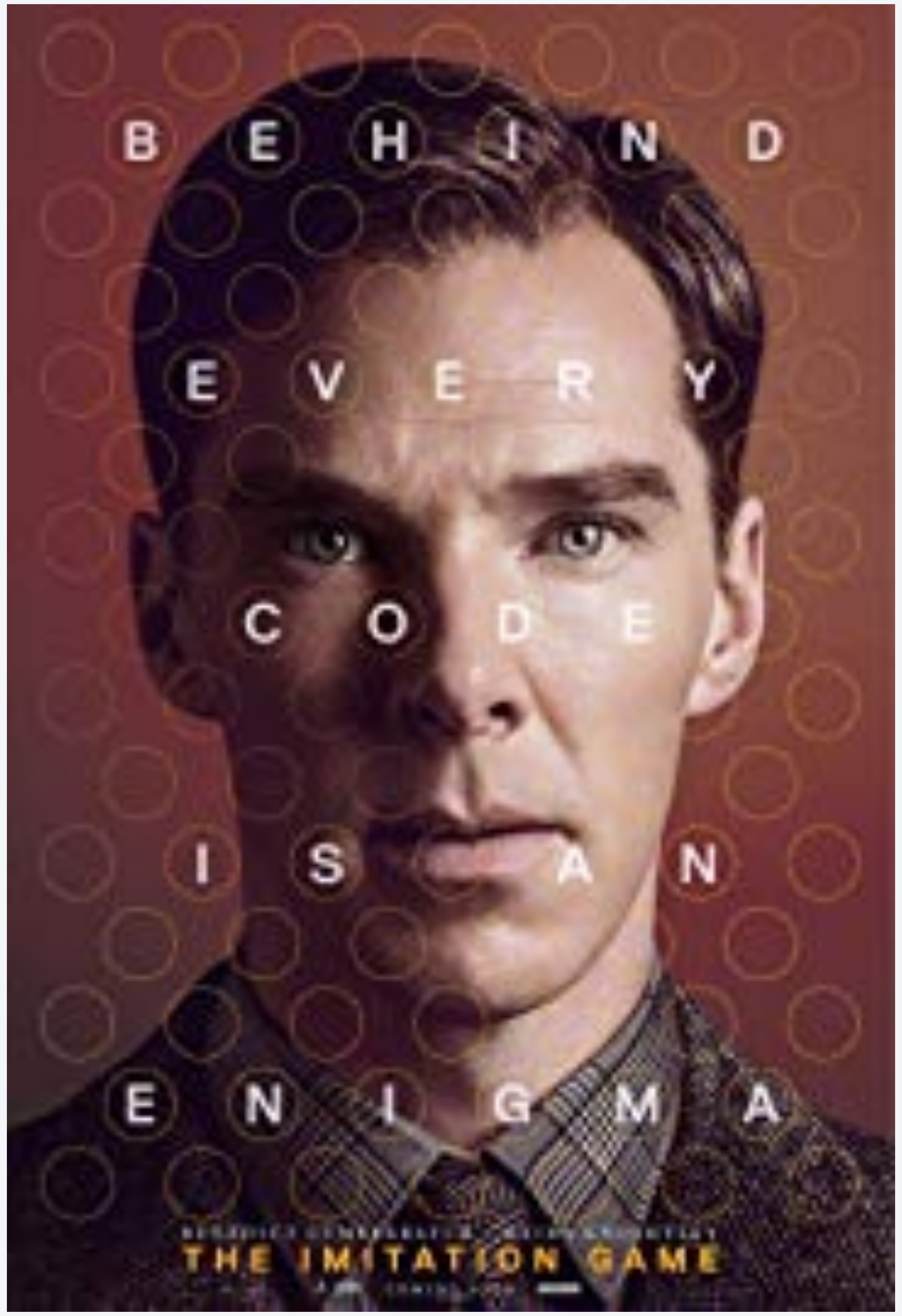
Enrichment II

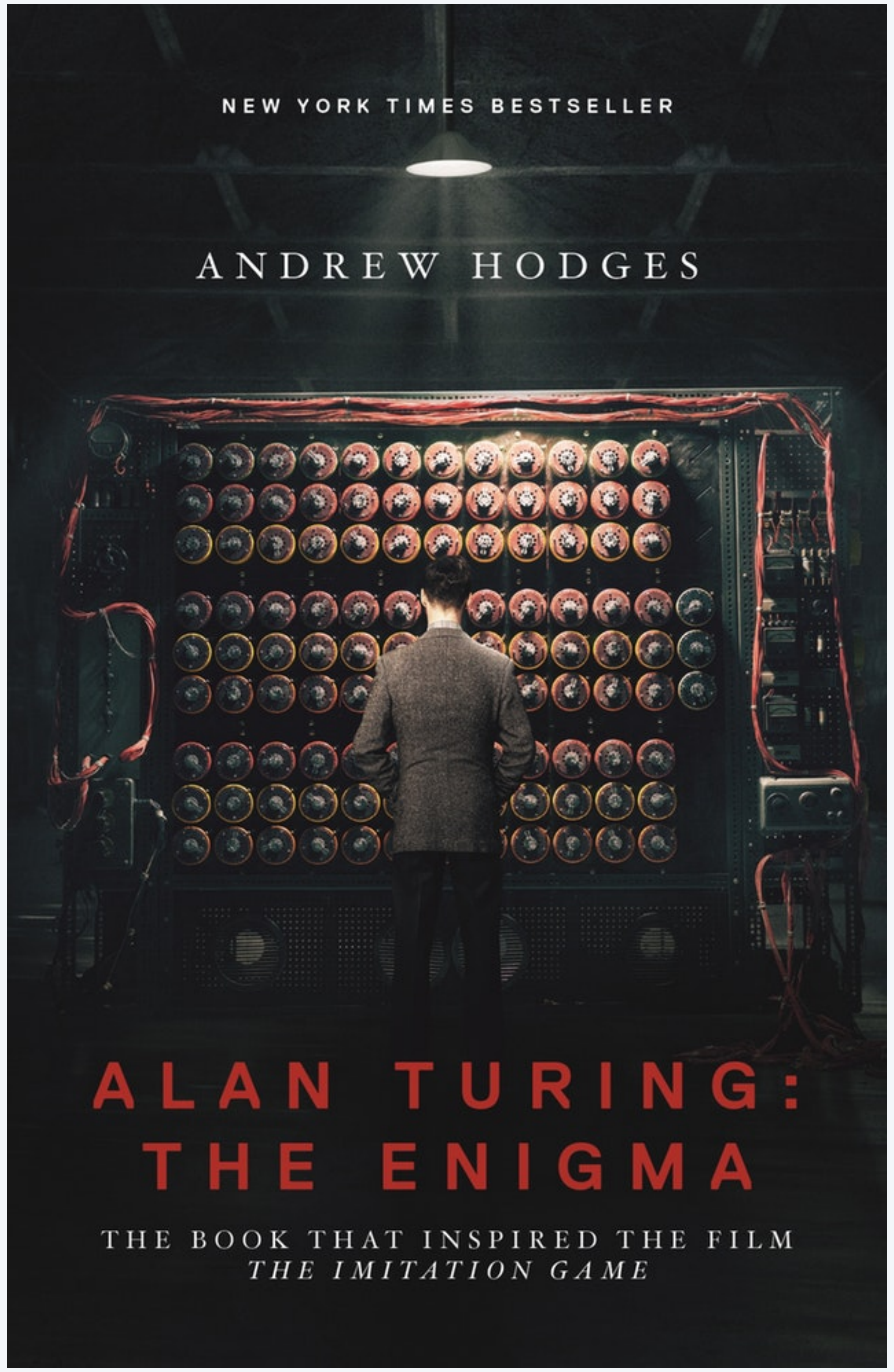




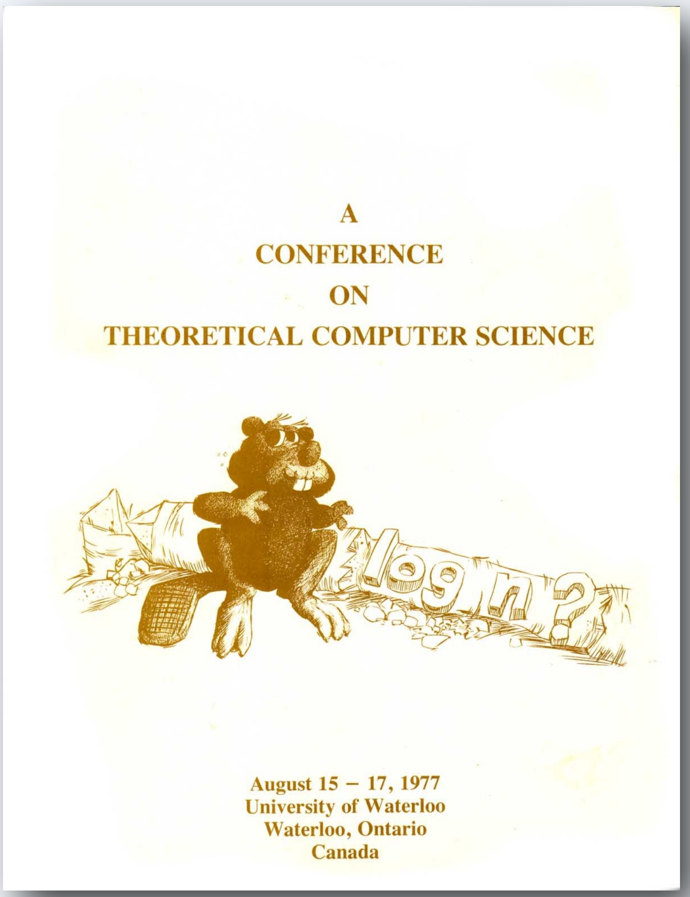
Enrichment III



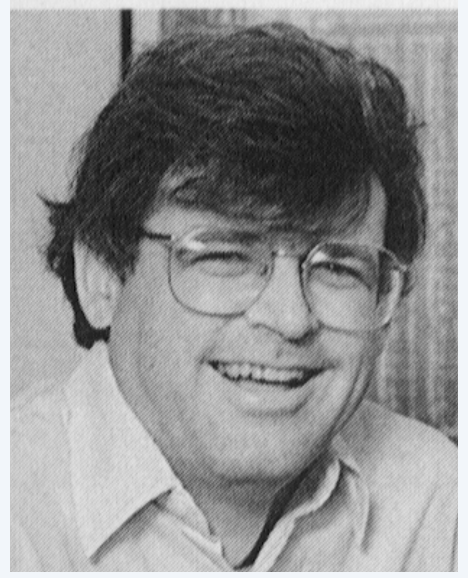




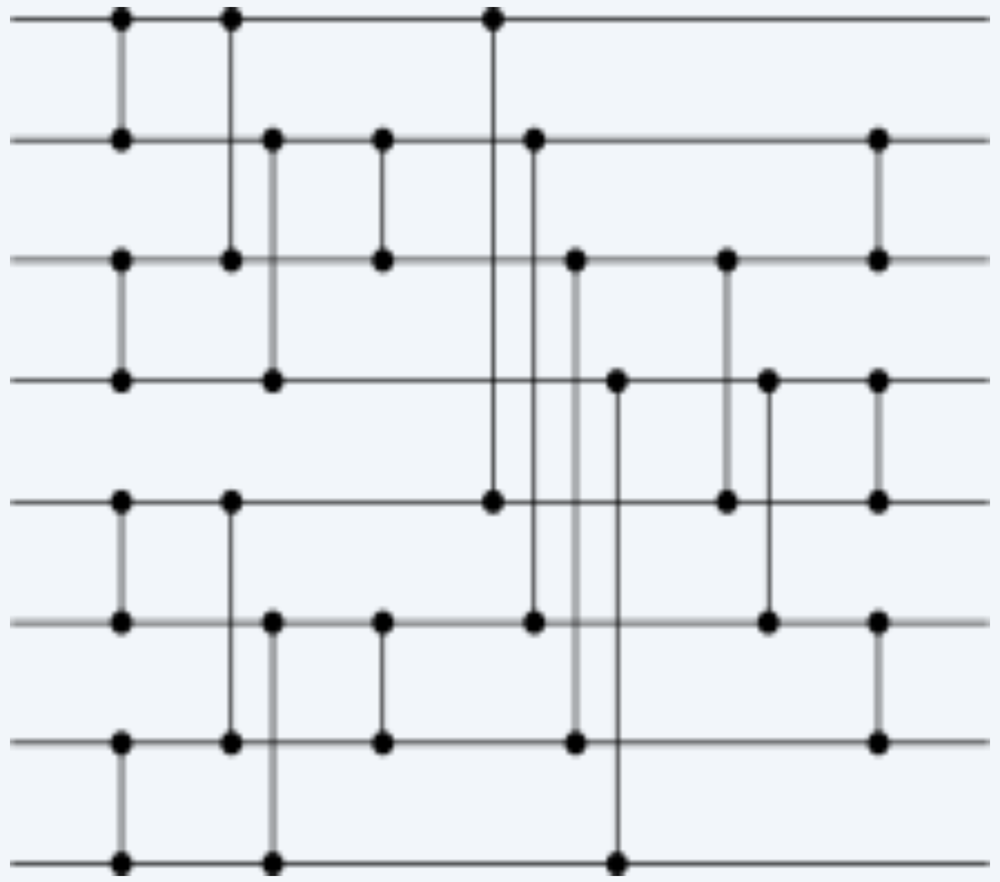
PF, 1977: "I believe that we have a formula in common!"

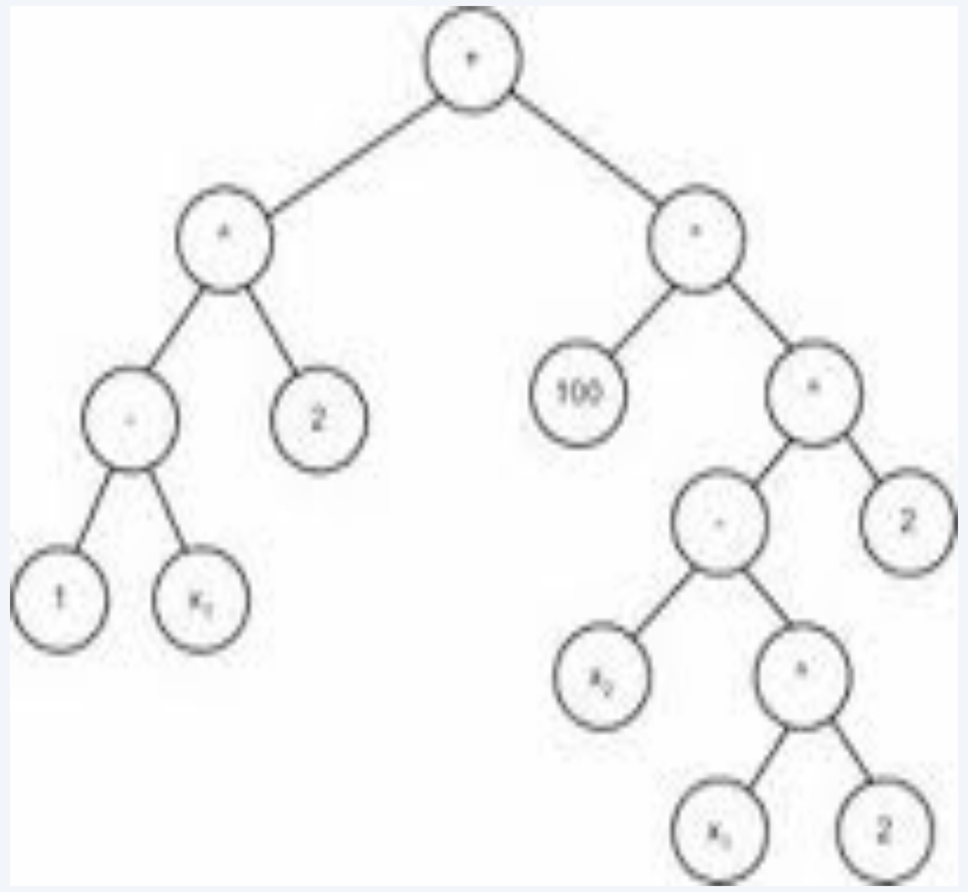


Data Movement in Odd-Even Merging
by Robert Sedgewick ....

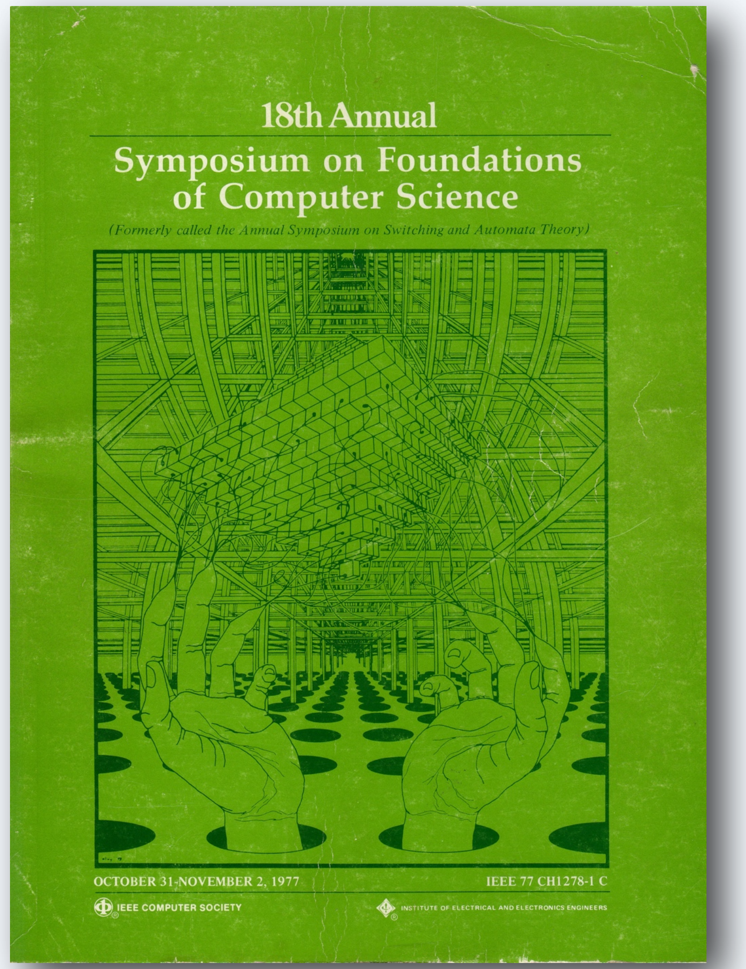


$$\frac{2}{\ln 2}\Gamma\left(\frac{k\pi i}{\ln 2}\right)\zeta\left(\frac{2k\pi i}{\ln 2},\frac{1}{4}\right)$$



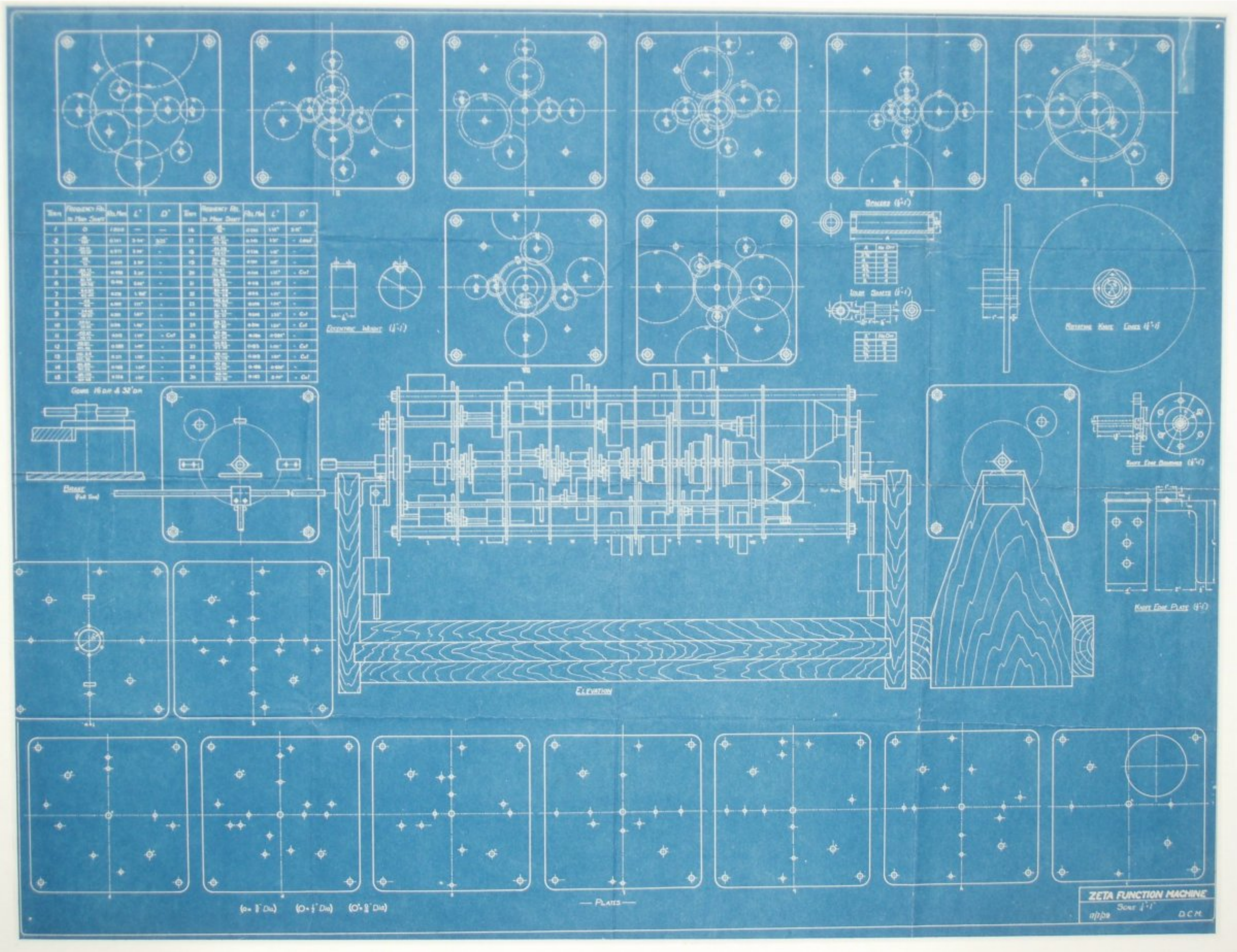


$$\frac{2k\pi i - \log 2}{\log 2}\Gamma\left(\frac{k\pi i}{\log 2}\right)\zeta\left(\frac{2k\pi i}{\log 2}\right)$$



On the Average Number of Registers Required for Evaluating Arithmetic Expressions
by P. Flajolet, J. C. Raoult, and J. Vuillemin ....

Turing's machine to calculate the zeta function



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