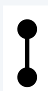
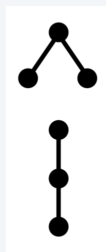
AofA Trees (or analytic combinatorics) Q&A 1

Def. A *unary-binary tree* is a rooted, ordered tree with node degrees all 0, 1, or 2.

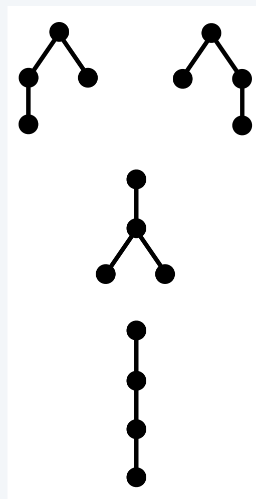
$M_1 = 1$



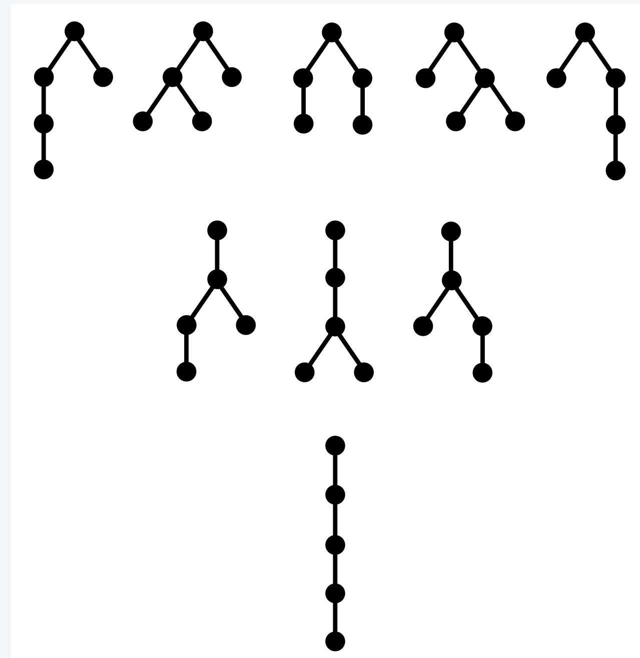
$M_2 = 1$



$M_3 = 2$



$M_4 = 4$



$M_5 = 9$

AofA Trees (or analytic combinatorics) Q&A 1

Q. How many unary-binary trees?

construction $$M = Z + Z \times M + Z \times M \times M$$

GF equation $$M(z) = z + zM(z) + zM(z)^2$$

explicit form
$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$
$$= \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}$$

coefficient asymptotics $$[z^n]M(z) = \frac{\sqrt{4/3}}{(4/3)\sqrt{\pi n^3}}3^n \quad = \frac{1}{\sqrt{4\pi n^3/3}}3^n$$

"Motzkin numbers"

$$[z^n]\frac{f(z)}{(1 - z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{-n}n^{\alpha-1}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$
$$\Gamma(1) = 1$$
$$\Gamma(s+1) = s\Gamma(s)$$

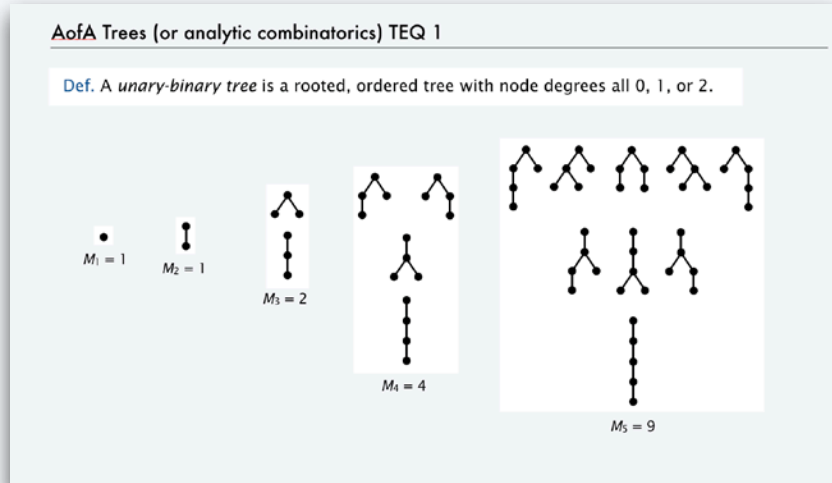
IMPORTANT NOTE: It's wise to check GF equations before trying to solve them!

$$M(z) = z + zM(z) + zM(z)^2$$

$$M(z) = z + z^2 + 2z^3 + 4z^4 + 9z^5 + \dots$$

$$\begin{aligned}
z &= z \\
zM(z) &= z^2 + z^3 + 2z^4 + 4z^5 + \dots \\
z(M(z)^2) &= z^3 + z^4 + 2z^5 + \dots \\
&\quad + z^4 + z^5 + \dots \\
&\quad + 2z^5 + \dots
\end{aligned}$$

$$z + zM(z) + zM(z)^2 = z + z^2 + 2z^3 + 4z^4 + 9z^5 + \dots$$



AofA Trees (or analytic combinatorics) Q&A 1 (improved version?)

Q. How many unary-binary trees?

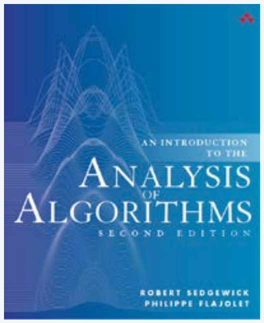
construction

GF equation

explicit form

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$

coefficient asymptotics



§6.15 TREES 335

Now, Theorem 4.11 provides an immediate proof that $[z^N]M(z)$ is $O(3^N)$, and methods from complex asymptotics yield the more accurate asymptotic estimate $3^N/\sqrt{3/4\pi N^3}$. Actually, with about the same amount of work, we can derive a much more general result.

Errata just posted:

- **335.** First sentence should read "The corollary to Theorem 5.5 (page 250) provides an immediate proof that $[z^n]M(z) \sim 3^n/\sqrt{4\pi n^3/3}$."

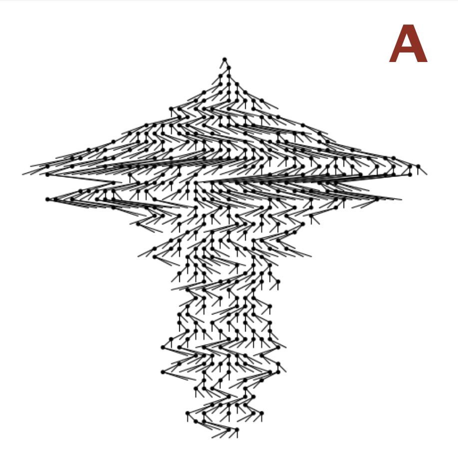
Complex asymptotics is often *not needed*.

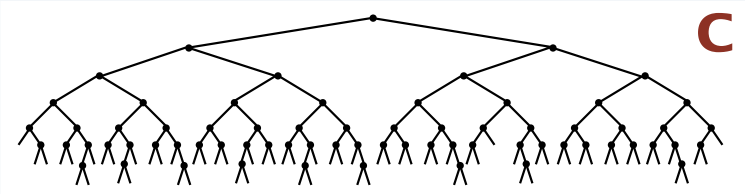
BUT it allows us to address entire classes of problems (stay tuned).

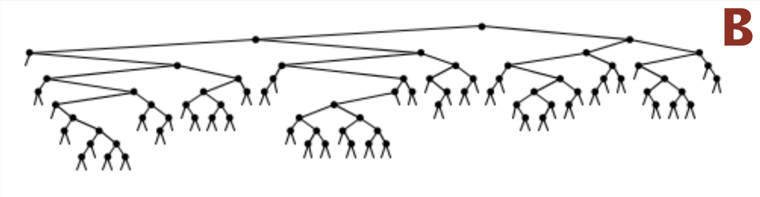
AofA Trees Q&A 2

Q. Match each diagram with its description

A.	random Catalan tree
B.	random BST
C.	random AVL tree



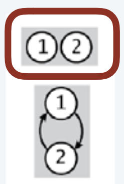




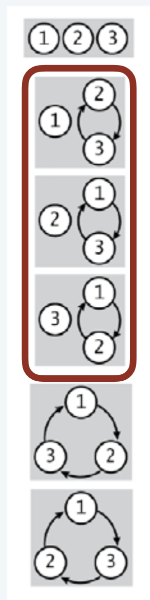
Some questions are *very easy* if you've watched the lecture (and impossible if not).

AofA permutations (or analytic combinatorics) Q&A 1

Q. What is the probability that a random perm of size n has exactly 2 cycles?



1/2



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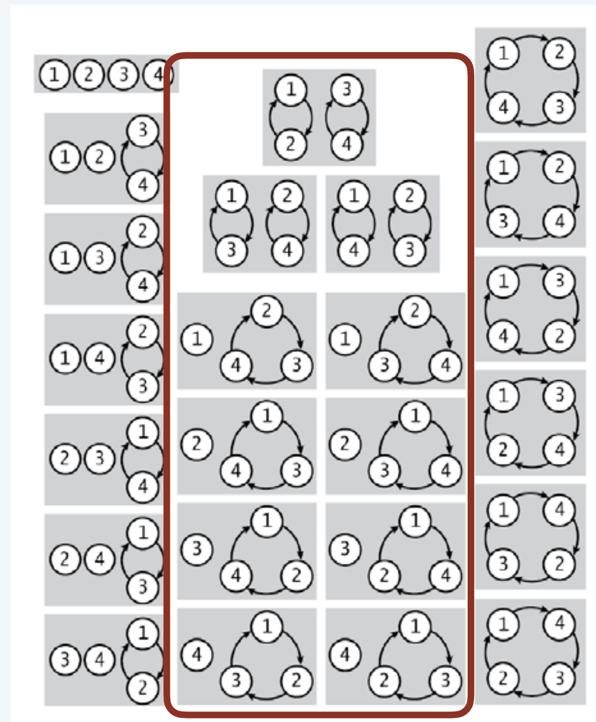

11/24

AofA Permutations (or analytic combinatorics) Q&A 1

Q. What is the probability that a random perm of size *n* has exactly 2 cycles?

construction

$$P_2 = SET_2(CYC(Z))$$

EGF equation

$$P(z) = \frac{1}{2}\Big(\ln\big(\frac{1}{1-z}\big)\Big)^2$$

coefficient asymptotics

$$\frac{1}{2}\ln\big(\frac{1}{1-z}\big)^2 = \sum_{n\ge 0} p_n \frac{z^n}{n!}$$

$$\frac{1}{1-z}\ln\big(\frac{1}{1-z}\big) = \sum_{n\ge 1} p_n \frac{z^{n-1}}{(n-1)!}$$

technique: differentiate both sides

$$p_n = (n-1)!H_{n-1}$$

A. H_{n-1}/n

AofA Permutations (or analytic combinatorics) Q&A 1 (improved version?)

Q. What is the probability that a random perm of size *n* has exactly 2 cycles?

construction

EGF equation

coefficient asymptotics

$$p_n = (n-1)!H_{n-1}$$

AofA Permutations Q&A 2

Q. Identify the most likely and least likely events for a random permutation of size 1000.

$$\frac{1}{e^{\alpha}}$$

$1 + 1/2 + 1/4 + 1/8 + \ldots 1/512 \doteq 1.998$	no cycles of length a power of 2	
$1 + 1/2 + 1/3 + 1/4 \doteq 2.0833$	no cycles of length less than 5	most likely
$1 + 1/2 + 1/3 + 1/7 \doteq 1.976$	no cycles of length 1, 2, 3, or 7	least likely
probably unfair	no cycles of length greater than 400	

100 prisoners solution

Problem. 100 prisoners, each uniquely identified by a numbered ID card (1 to 100), have been sentenced to death, but are given a last chance.

- The ID cards are collected and put in the drawers of a cabinet with 100 numbered drawers (1 to 100) in random order, one card per drawer.
- One at a time, the prisoners are allowed to enter the room containing the cabinet and open, then close again, at most *half* the drawers.
- If *all* prisoners find their own number, they will all be spared.
- If *one prisoner fails*, they will all be executed.

Prisoner B's strategy: Each prisoner "follows the cycle"

- Opens the drawer corresponding to his ID.
- Uses the number in that drawer to decide which drawer to open next.
- Continues until finding the drawer containing his ID.

Q. When does Prisoner's B strategy succeed?

A. When the random permutation has no cycles of length greater than 50.

Probability of success: $[z^{100}]\exp\left(\frac{z}{1}+\frac{z}{2}+\ldots+\frac{z}{50}\right) = 1-(H_{100}-H_{50}) \doteq 0.31$

$$[z^{1000}]\exp\left(\frac{z}{1}+\frac{z^2}{2}+\ldots+\frac{z^{400}}{400}\right) = 1-(H_{1000}-H_{400})$$