# AofA Analytic Combinatorics Q&A 0

Q. How many binary strings of length N have no occurrence of 01?

A. Trick question!

# AofA Analytic Combinatorics Q&A 1

Q. How many binary strings of length N have no occurrence of 011?

A.

### Symbolic method: binary strings with restrictions

#### Ex. How many N-bit binary strings have no two consecutive 0s?

Class	$B_{00}$ , the class of binary strings with no 00
Size	b , the number of bits in $b$
OGF	$B_{00}(z) = \sum_{b \in B_{00}} z^{ b }$

type class size GF 0 bit 1 bit

**Atoms** 

Construction 
$$B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$$

"a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00"

$$B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$$

$$B_{00}(z) = \frac{1+z}{1-z-z^2}$$

$$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$$
 1, 1, 2, 3, 5, 8, 13, ...

### Probably need a 1-page cheatsheet.



$$B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$$

$$B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$$

$$B_{00}(z) = \frac{1+z}{1-z-z^2}$$

B<sub>011</sub>: class of binary strings with no 011

P<sub>011</sub>: class of binary strings whose only 011 is at the beginning

EQ 1 
$$E + (Z_0 + Z_1) \times B_{011} = B_{011} + P_{011}$$
  
  $1 + 2zB_{011}(z) = B_{011}(z) + P_{011}(z)$ 

Note: true for any pattern p: prepending a 0 or a 1 to a p-free binary string gives a p-free binary string or a binary string whose only occurrence of p is at the beginning"

EQ 2 
$$(Z_0 \times Z_1 \times Z_1) \times B_{011} = P_{011}$$
 
$$z^3 B_{011}(z) = P_{011}(z)$$

Note: In general, RHS may be more complicated

$$B_{001}(z) = \frac{1}{1 - 2z + z^3} = \frac{1}{(1 - z)(1 - z - z^2)} = \frac{1}{(1 - z)(1 - \phi z)(1 - \hat{\phi}z)}$$

Probably too intricate for an inclass midterm (borderline, at best)

$$[z^n]B_{001}(z) = \frac{\phi^n}{\sqrt{5}}$$

B<sub>010</sub>: class of binary strings with no 010

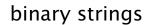
P<sub>010</sub>: class of binary strings whose only 010 is at the beginning

EQ 1 
$$E + (Z_0 + Z_1) \times B_{010} = B_{010} + P_{010}$$
  
EQ 2  $(Z_0 \times Z_1 \times Z_0) \times B_{010} = P_{010} + (Z_0 \times Z_1) \times P_{010}$ 

if the string from  $B_{010}$  begins with 10 constructed string begins with 01010 (two occurrences of 010) 01 followed by a string whose only 010 is at the beginning

## AofA Analytic Combinatorics Q&A 1 (improved version)

Q. Match each combinatorial class with a construction.



binary strings with no 01

binary strings with no 11

binary strings with no 001

binary strings with no 00

$$E + (Z_0 + Z_1) \times B = B + (Z_0 \times Z_1) \times B$$

$$B = E + Z_1 + (Z_0 + Z_1 \times Z_0) \times B$$

$$E + B \times (Z_0 + Z_1) = B + B \times (Z_0 \times Z_0 \times Z_1)$$

$$E + B \times (Z_0 \times Z_1) = B + (Z_0 \times (Z_0 + Z_1)) \times B$$

$$B = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B$$

$$B = E + (Z_0 + Z_1) \times B$$

## AofA Analytic Combinatorics Q&A 2 (2015 exam)

**Q6.** Let 
$$k_n = [z^n] \frac{1}{\sqrt{1-3z}} \log \frac{1}{1-2z}$$
. Then:

O 
$$k_n \sim \log 3 \frac{3^n}{\sqrt{\pi n}}$$

O 
$$k_n \sim \sqrt{2} \frac{2^n}{n}$$

Lesson: Need a 1-page cheatsheet.

with correct version of formula!

$$[z^n] \frac{f(z)}{(1 - z/\rho)^{\alpha}} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^{-n} n^{\alpha - 1}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(s + 1) = s\Gamma(s)$$

# AofA Analytic Combinatorics Q&A 2 (improved version)

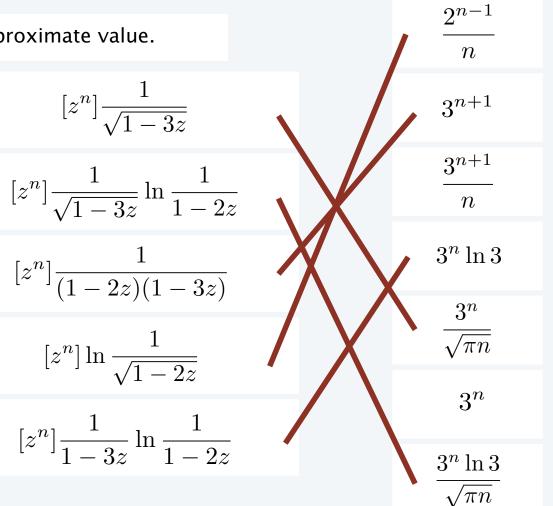
Q. Match each expression with an approximate value.

$$[z^n] \frac{f(z)}{(1 - z/\rho)^{\alpha}} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^{-n} n^{\alpha - 1}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(s + 1) = s\Gamma(s)$$



The hard way:

$$[z^n] \frac{1}{(1-2z)(1-3z)} = [z^n] \left(\sum_{n\geq 0} 2^n z^n\right) \left(\sum_{n\geq 0} 3^n z^n\right)$$
$$= \sum_{0\leq k\leq n} 2^k 3^{n-k}$$
$$= 3^n \sum_{0\leq k\leq n} (2/3)^k$$

Still not convinced?

