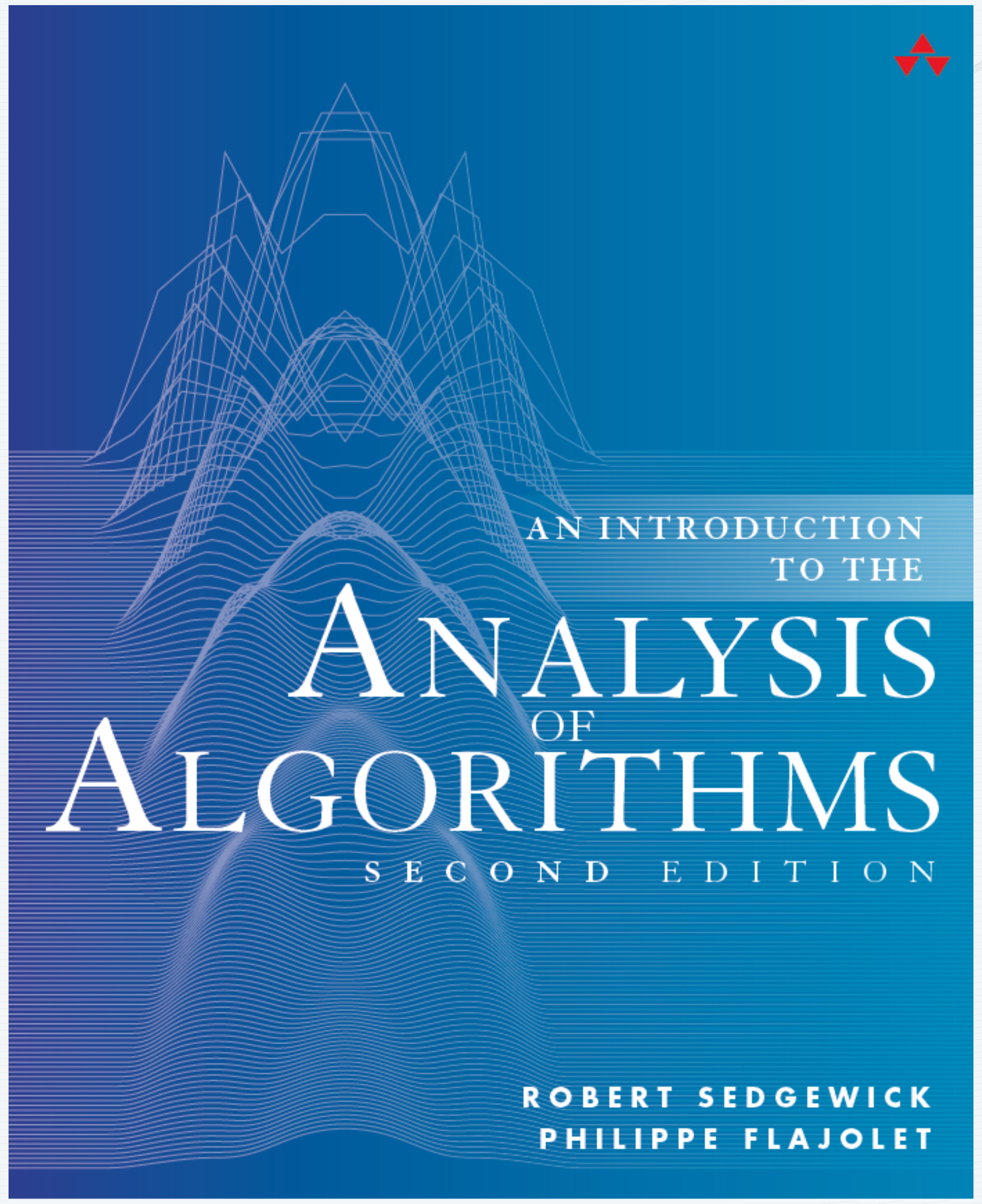
An Introduction to
ANALYTIC COMBINATORICS

Computer Science 488
Robert Sedgewick



http://aofa.cs.princeton.edu

AofA REVIEW

Basic facts about review problem sets

The AofA review problem set will be posted at 11:59AM 3/16 due at 11:59PM 3/17.

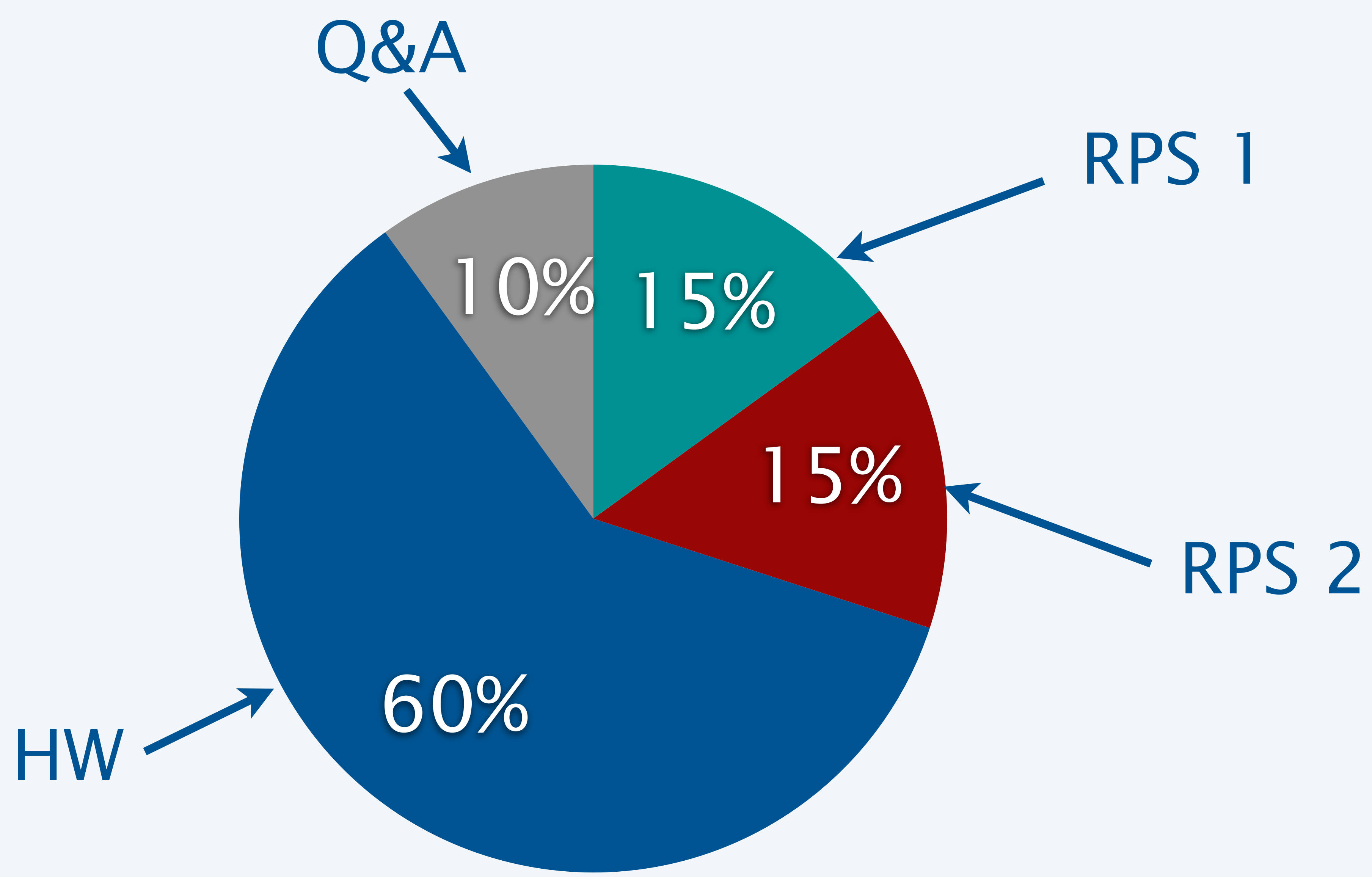
Purpose

- Check understanding of key facts.
- Build confidence in what you have learned.
- Prepare for the next level.

We know that you don't have much time to study

- Questions are like old exam questions, adjusted because you have more time to solve them.
- At most 1 question per lecture
- Review the material and your own work.
- Create a "cheatsheet" of important facts.

Each problem is a small part (<2%) of the story.



AofA Chapter 1 Intro Q&A example

Q. Solve the following recurrence

$$F_N = N^2 + \frac{1}{N}\sum_{1\le k\le N}(F_{k-1} + F_{N-k}) \quad \text{with} \quad F_0 = 0.$$

$$C_N = N + 1 + \frac{1}{N}\sum_{1\le j\le N}(C_{j-1} + C_{N-j})$$

$$= N + 1 + \frac{2}{N}\sum_{1\le j\le N} C_{j-1}$$

$$NC_N - (N-1)C_{N-1} = 2N + 2C_{N-1}$$

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

$$\ldots$$

Maybe put summary of Quicksort derivation on your cheatsheet.

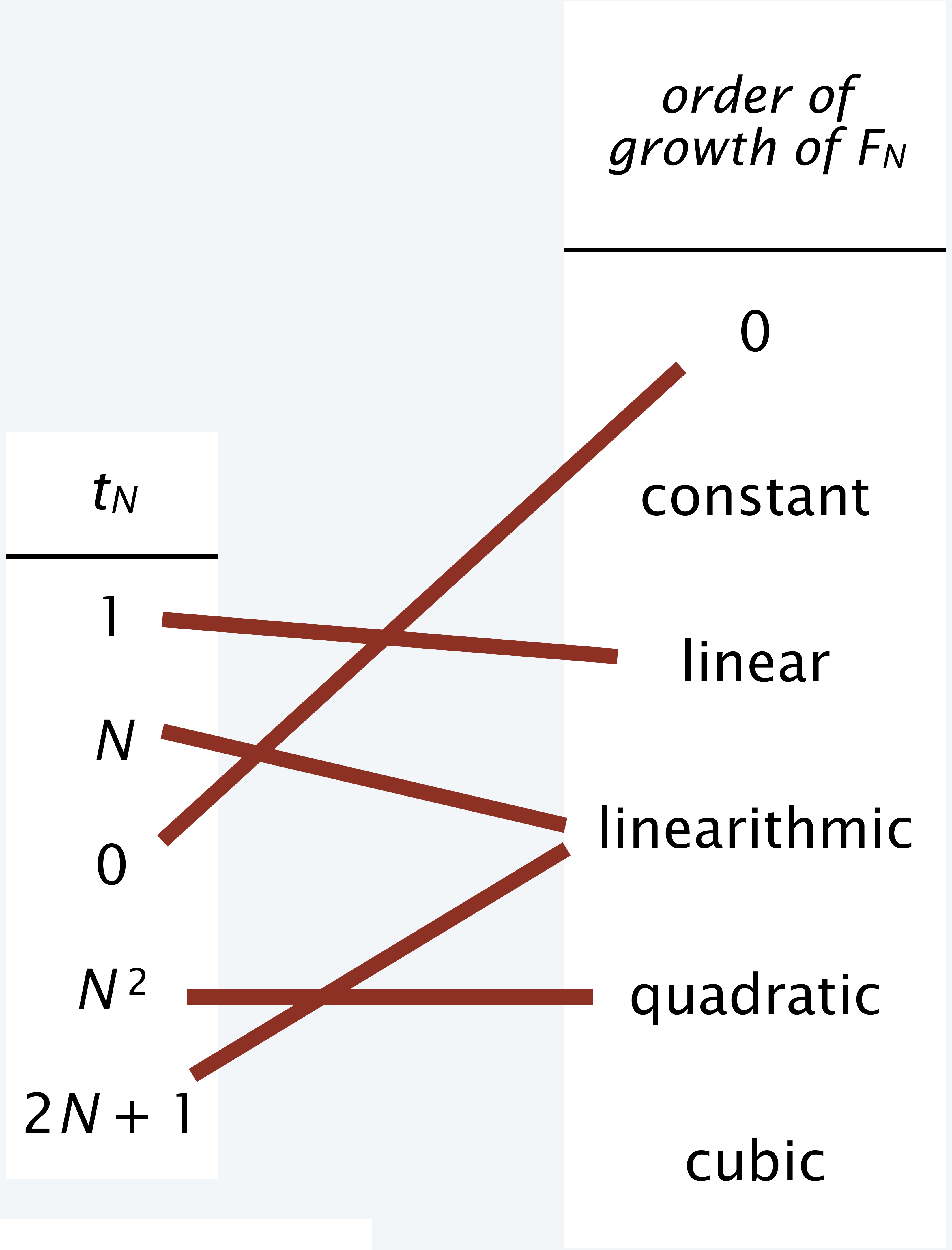
AofA Chapter 1 Intro Q&A example (revised)

Q. Match each "toll function" at left with the order of growth of the solution at right for the Quicksort recurrence

$$F_N = t_N + \frac{1}{N} \sum_{1 \le k \le N} (F_{k-1} + F_{N-k}) \quad \text{with} \quad F_0 = 0$$

$$NC_N - (N+1)C_{N-1} = 3N^2 - 3N + 1$$

$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + 3 + \ldots$$



A good question *must* avoid answers that depend on detailed calculations.

AofA Chapter 2 Recurrences Q&A example

Q. Solve the recurrence

$$na_n = (n-3)a_{n-1} + n \quad \text{for} \quad n \geq 3 \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \leq 2$$

n	a_n
3	1
4	5/4
5	3/2
6	7/4

summation factor

A.

$$\overline{n(n-1)(n-2)}a_n = \overline{(n-1)(n-2)}(n-3)a_{n-1} + \overline{n(n-1)(n-2)} \quad \text{for} \quad n \geq 3$$

$$\binom{n}{3}a_n = \binom{n-1}{3}a_{n-1} + \binom{n}{3} \quad \text{for} \quad n \geq 3$$

$$= \sum_{3 \leq k \leq n} \binom{k}{3} \quad = \binom{n+1}{4}$$

$$a_n = \frac{n+1}{4}$$

Borderline suitable for a COS 488 inclass exam, but OK for a review problem.

easier: $na_n = (n-3)a_{n-1} + 4n$

AofA Chapter 3 GFs Q&A example

Q. Match each of sequence with its OGF.

0, 0, 1, 3, 6, 10, ...	$\frac{1}{1-3z}$
0, 0, 1/2, 0, 1/4, 0, 1/6, ...	$\frac{z^2}{(1-z)^3}$
1, 3, 9, 27, 81, 243, ...	$\ln\frac{1}{1-z^2}$
1, 1 + 1/2, 1 + 1/2 + 1/3, ...	$\frac{3}{1-z}$
3, 3, 3, 3, 3, ...	$\ln\frac{1}{1-2z}$
	$\frac{1}{1-z}\ln\frac{1}{1-z}$
	$\frac{1}{(1-z)^3}$

$$\frac{z^M}{(1-z)^M} = \sum_{N\geq M}\binom{N}{M}z^M$$

$$\ln\frac{1}{1-z} = \sum_{N\geq 1}\frac{z^N}{N}$$

Maybe put formulas you won't quickly remember on the cheatsheet.

OK for a COS 488 inclass exam, but too easy for a review problem.

AofA Chapter 4 Asymptotics Q&A example 1

Q. Give an asymptotic approximation of $e^{H_{2N}-H_N}$ to within $O(\frac{1}{N^2})$

A.

$$H_{2N} - H_N = \ln(2N) + \gamma + \frac{1}{4N} + O\Big(\frac{1}{N^2}\Big)$$
$$- \ln N - \gamma - \frac{1}{2N} + O\Big(\frac{1}{N^2}\Big)$$
$$= \ln 2 - \frac{1}{4N} + O\Big(\frac{1}{N^2}\Big)$$

$$\exp\Big(\ln 2 - \frac{1}{4N}\Big) = 2\exp\Big(-\frac{1}{4N}\Big) = \boxed{2 - \frac{1}{2N} + O\Big(\frac{1}{N^2}\Big)}$$

$$H_N = \ln N + \gamma + \frac{1}{2N} + O\Big(\frac{1}{N^2}\Big)$$

$$e^x = 1 + x + \frac{x^2}{2} + O\big(x^3\big)$$

Maybe put formulas you won't quickly remember on the cheatsheet.

AofA Chapter 4 Asymptotics Q&A example 1 (improved version)

Q. Match each function with an asymptotic expansion.

H_N	$1 + \frac{1}{2N} + O\left(\frac{1}{N^2}\right)$
$\exp\left(H_{2N} - H_N\right) - 1$	$1 + \frac{1}{N} + O\left(\frac{1}{N^2}\right)$
$\exp\left(H_N\right)$	$N + O(1)$
$\exp\left(\frac{1}{N}\right)$	$1 - \frac{1}{N} + O\left(\frac{1}{N^2}\right)$
$\left(1 + \frac{1}{N}\right)^{-1}$	$1 - \frac{1}{2N} + O\left(\frac{1}{N^2}\right)$
	$\ln N + \gamma + \frac{1}{2N} + O\left(\frac{1}{N^2}\right)$
	$N + \gamma + O\left(\frac{1}{N}\right)$

AofA Chapter 4 Asymptotics Q&A Example 2

Q. Match each of the topics described in the book with a mathematician's name.

- Approximate a sum with an integral
- Expand a differentiable function
- Approximate factorials
- Birthday function
- Approximate a function by swapping tails

- Laplace
- Ramanujan
- Euler
- Taylor
- Stirling

No, this is not high school, but... *You do not want to appear to be ignorant*!

Q. Match each expression with an approximation to its value.

1.01^{10}
1.05^{10}
1.01^{20}
1.01^{50}
1.01^{100}

1.10102
1.10462
1.22019
1.50034
1.62889
1.64463
2.02300
2.70481
2.71828

$$(1+x)^t = \sum_{0\le k\le t} \binom{t}{k} x^k$$

$$= 1 + tx + \frac{t(t-1)}{2}x^2 + O(x^3)$$

$$(1+\frac{1}{N})^t = 1 + \frac{t}{N} + \frac{t(t-1)}{2N^2} + O(\frac{1}{N^3})$$

$$1.01^{10} = 1 + \frac{10}{100} + \frac{90}{20000} + \ldots$$

$$\approx 1.1045$$

$$(1+\frac{1}{N})^{\alpha N} = 1 + \frac{\alpha N}{N} + \frac{\alpha^2 N^2}{2N^2} + \ldots \text{✗}$$

$$(1+\frac{1}{N})^{\alpha N} = \exp\bigl(\alpha N \ln(1+1/N)\bigr)$$

$$= \exp\bigl(\alpha N(1/N + O(1/N^2))\bigr)$$

$$= e^{\alpha} + O(\frac{1}{N})$$

$$1.01^{50} \approx \sqrt{e}$$

AofA Chapter 4 Asymptotics Q&A Example 3

Q. Match each expression with an approximation to its value.

1.01^{10}

1.05^{10}

1.01^{20}

1.01^{50}

1.01^{100}

1.10102

1.10462

1.22019

1.50034

1.62889

1.64463

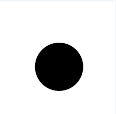
2.02300

2.70481

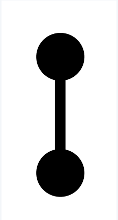
2.71828

Great for a COS 488 inclass exam, but no good for a review problem.

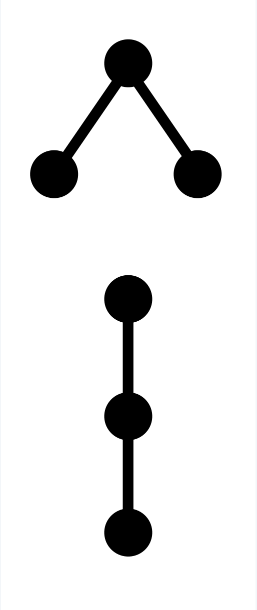
Def. A *unary-binary tree* is a rooted, ordered tree with node degrees all 0, 1, or 2.



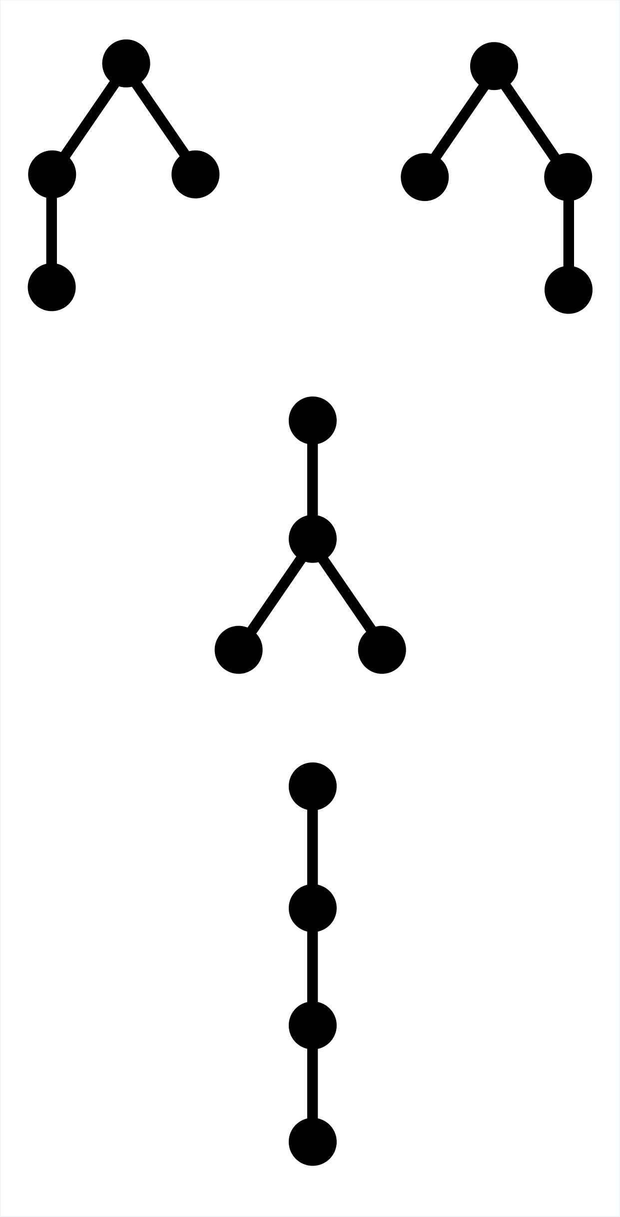
$M_1 = 1$



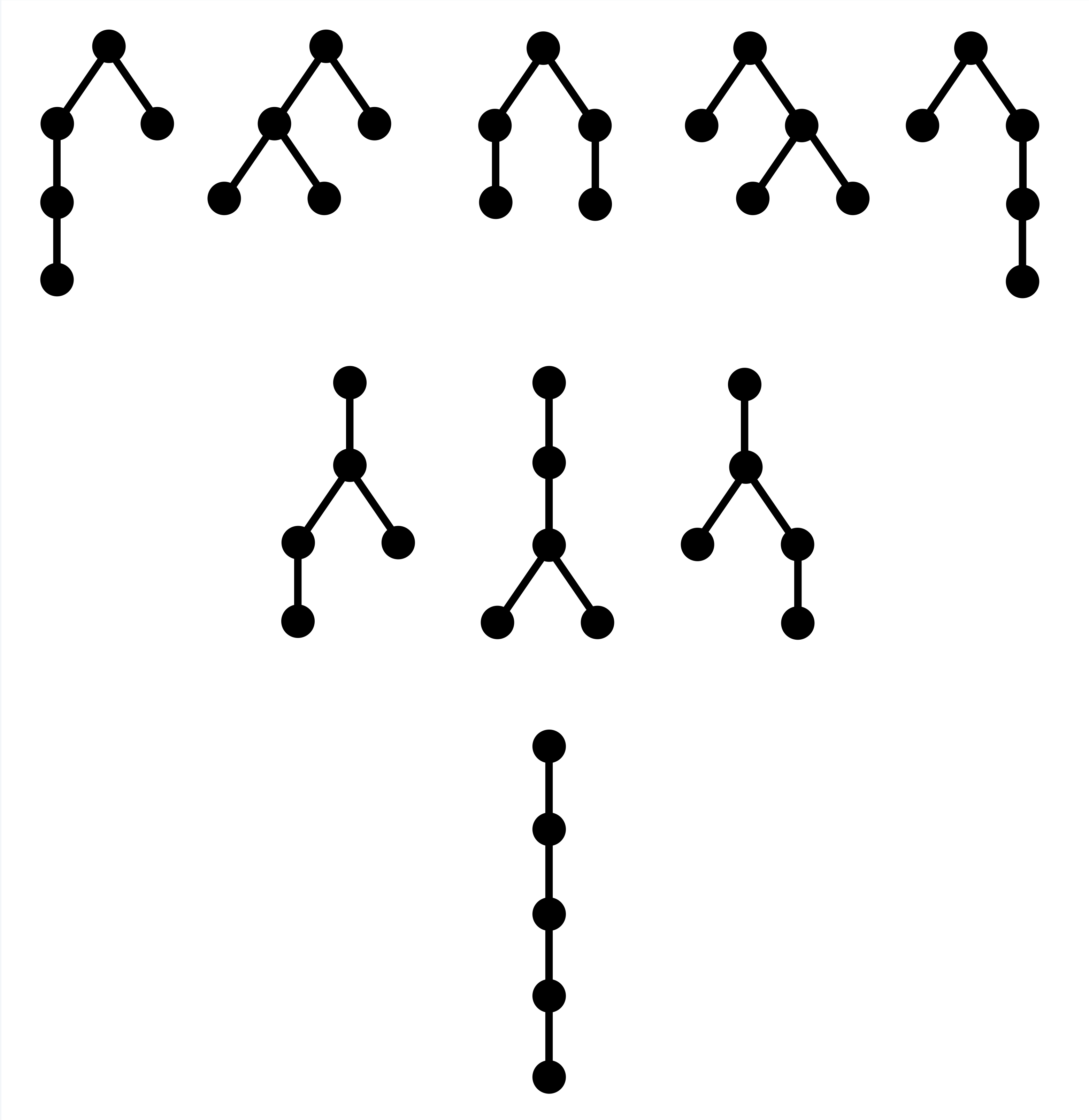
$M_2 = 1$



$M_3 = 2$



$M_4 = 4$



$M_5 = 9$

AofA Chapter 5 Analytic Combinatorics (or Trees) Q&A example

Q. How many unary-binary trees?

construction $$M = Z + Z \times M + Z \times M \times M$$

GF equation $$M(z) = z + zM(z) + zM(z)^2$$

explicit form $$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$

$$= \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}$$

coefficient asymptotics $$[z^n]M(z) = \frac{\sqrt{4/3}}{(4/3)\sqrt{\pi n^3}}3^n \quad = \frac{1}{\sqrt{4\pi n^3/3}}3^n$$

"Motzkin numbers"

$$[z^n]\frac{f(z)}{(1 - z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{-n}n^{\alpha-1}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(s+1) = s\Gamma(s)$$

✗ NOT AN EXAM QUESTION (too much calculation)

But OK for a review problem set

IMPORTANT NOTE: It's wise to check GF equations before trying to solve them!

$$M(z) = z + zM(z) + zM(z)^2$$

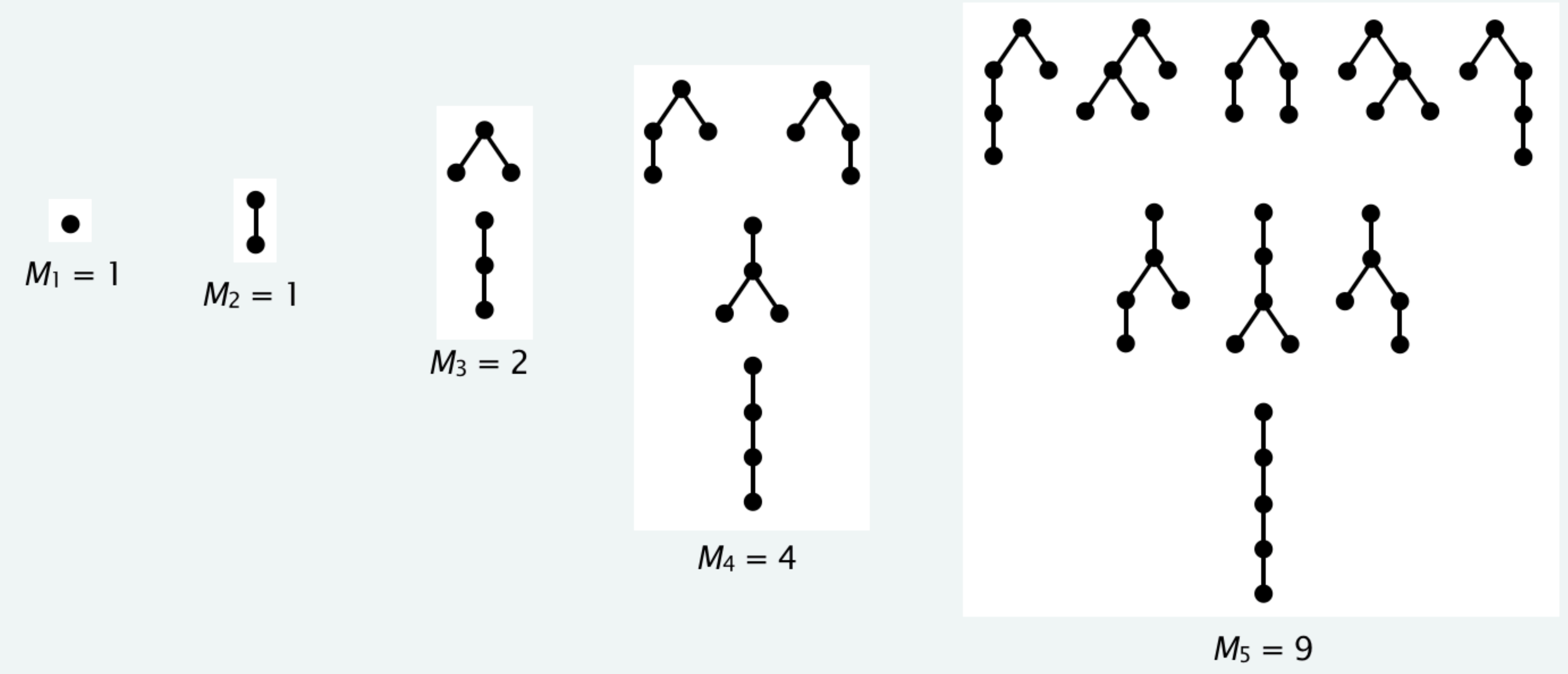
$$M(z) = z + z^2 + 2z^3 + 4z^4 + 9z^5 + \ldots$$

$$\begin{aligned} z &= z \\ zM(z) &= \quad z^2 + z^3 + 2z^4 + 4z^5 + \ldots \\ z(M(z)^2) &= \qquad z^3 + z^4 + 2z^5 + \ldots \\ & \qquad\qquad +z^4 + z^5 + \ldots \\ & \qquad\qquad\qquad +2z^5 + \ldots \end{aligned}$$

$$z + zM(z) + zM(z)^2 = z + z^2 + 2z^3 + 4z^4 + 9z^5 + \ldots$$

AofA Trees (or analytic combinatorics) TEQ 1

Def. A *unary-binary tree* is a rooted, ordered tree with node degrees all 0, 1, or 2.



AofA Chapter 5 Analytic Combinatorics (or Trees) Q&A example (improved version)

Q. How many unary-binary trees?

construction

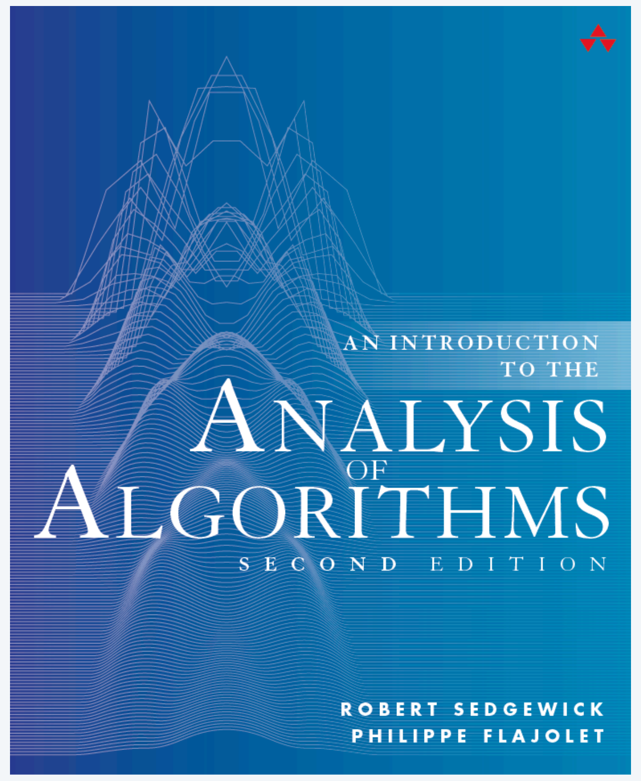
GF equation

explicit form

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$

coefficient asymptotics

Aside



§6.15 TREES 335

Now, Theorem 4.11 provides an immediate proof that $[z^N]M(z)$ is $O(3^N)$, and methods from complex asymptotics yield the more accurate asymptotic estimate $3^N/\sqrt{3/4\pi N^3}$. Actually, with about the same amount of work, we can derive a much more general result.

Errata posted in 2019:

- **335.** First sentence should read "The corollary to Theorem 5.5 (page 250) provides an immediate proof that $[z^n]M(z) \sim 3^n/\sqrt{4\pi n^3/3}$."

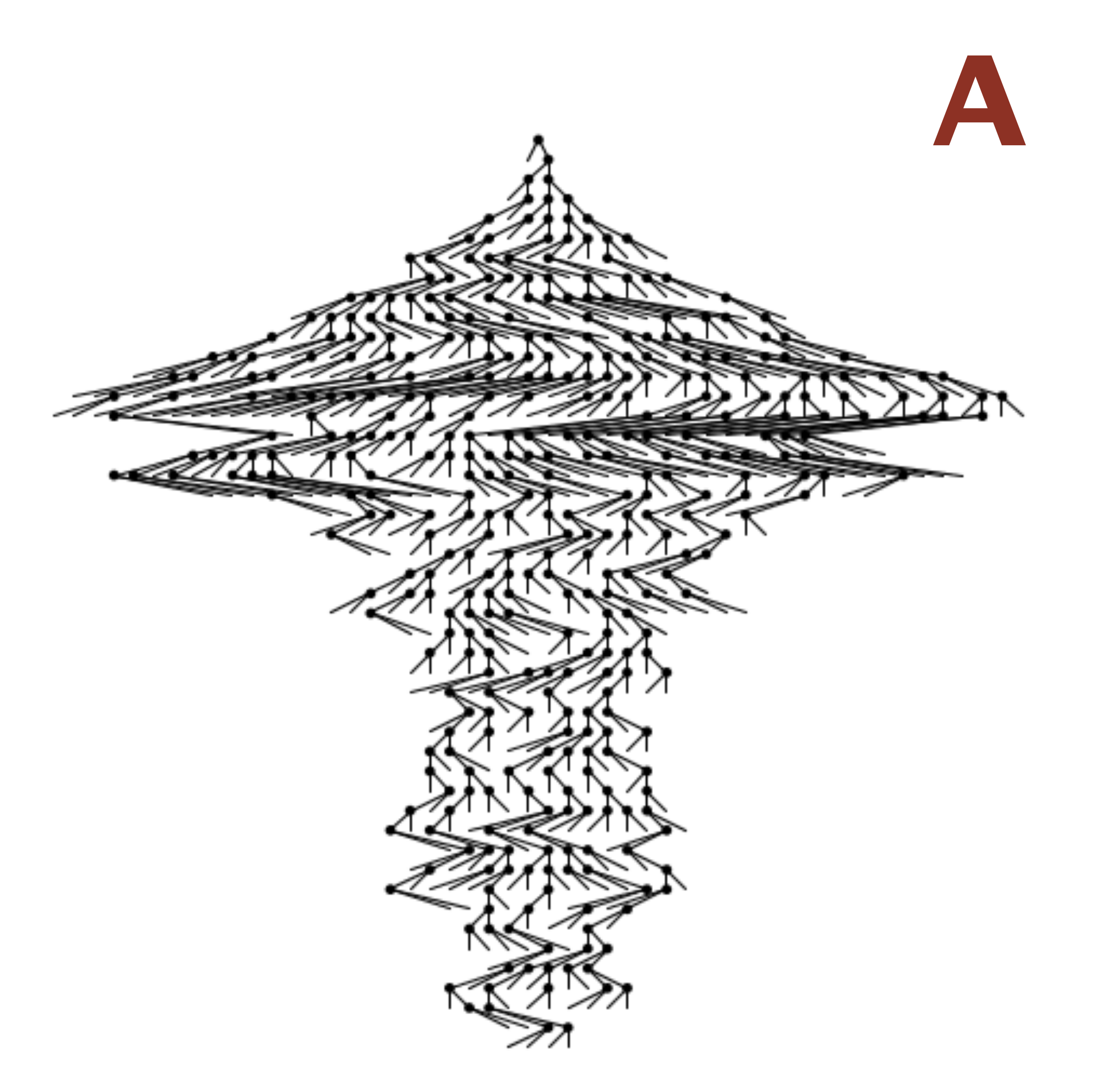
Complex asymptotics is often *not needed.*

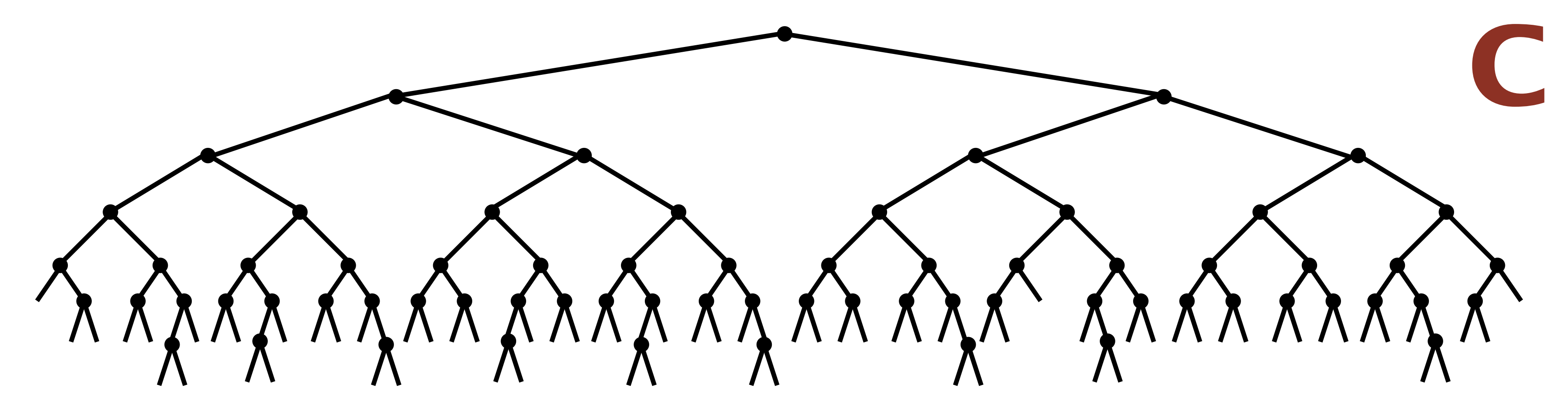
BUT it allows us to address entire classes of problems (stay tuned).

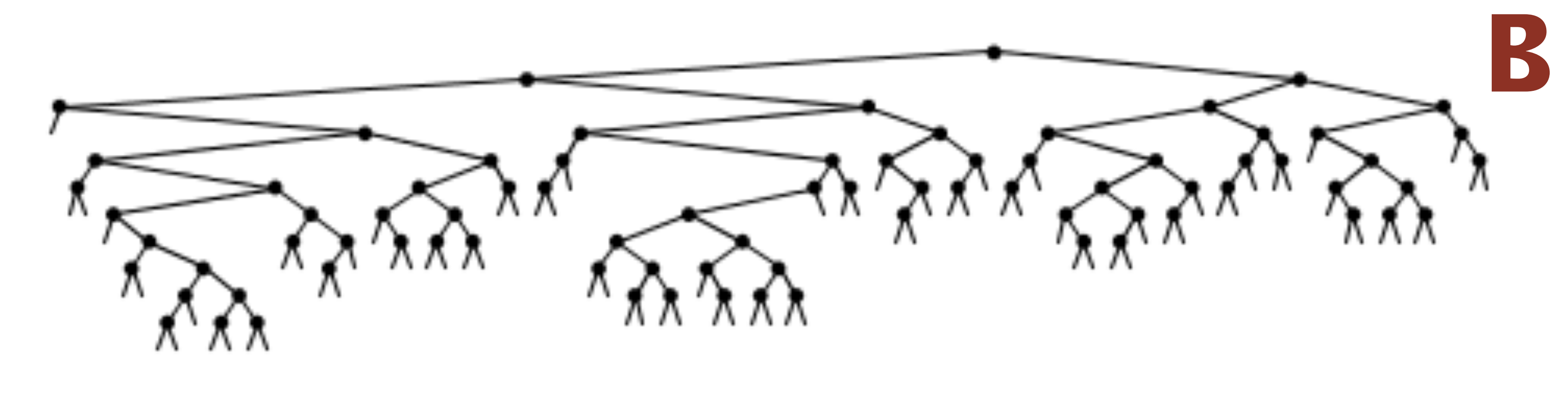
AofA Chapter 5 Trees Q&A example

Q. Match each diagram with its description

A.	random Catalan tree
B.	random BST
C.	random AVL tree





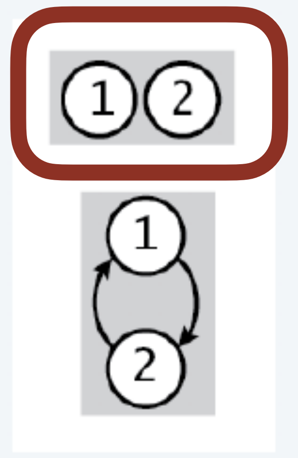


Some questions are *very easy* if you've watched the lecture (and impossible if not).

Too easy for a review problem.

AofA Chapter 7 Permutations (or analytic combinatorics) Q&A example

Q. What is the probability that a random perm of size *n* has exactly 2 cycles?



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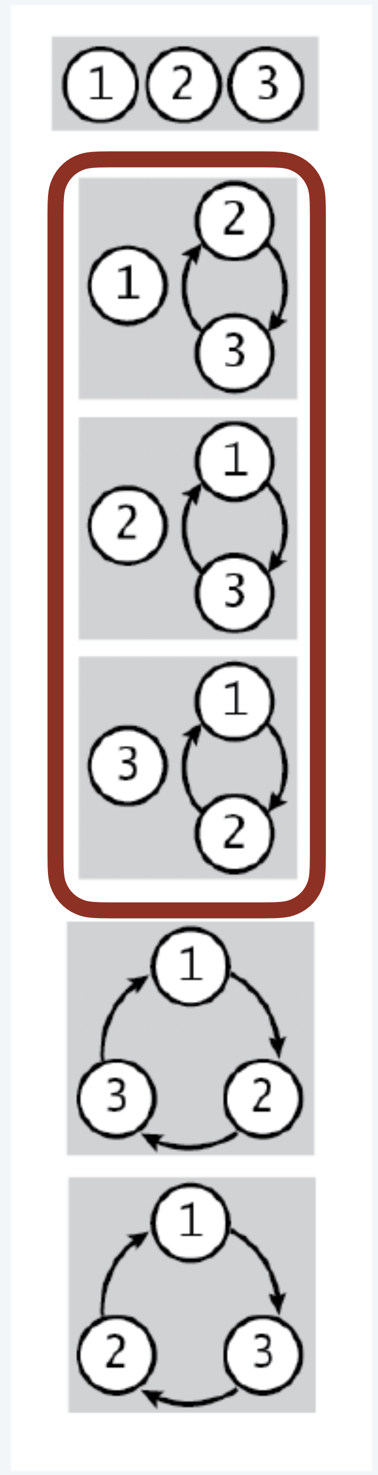

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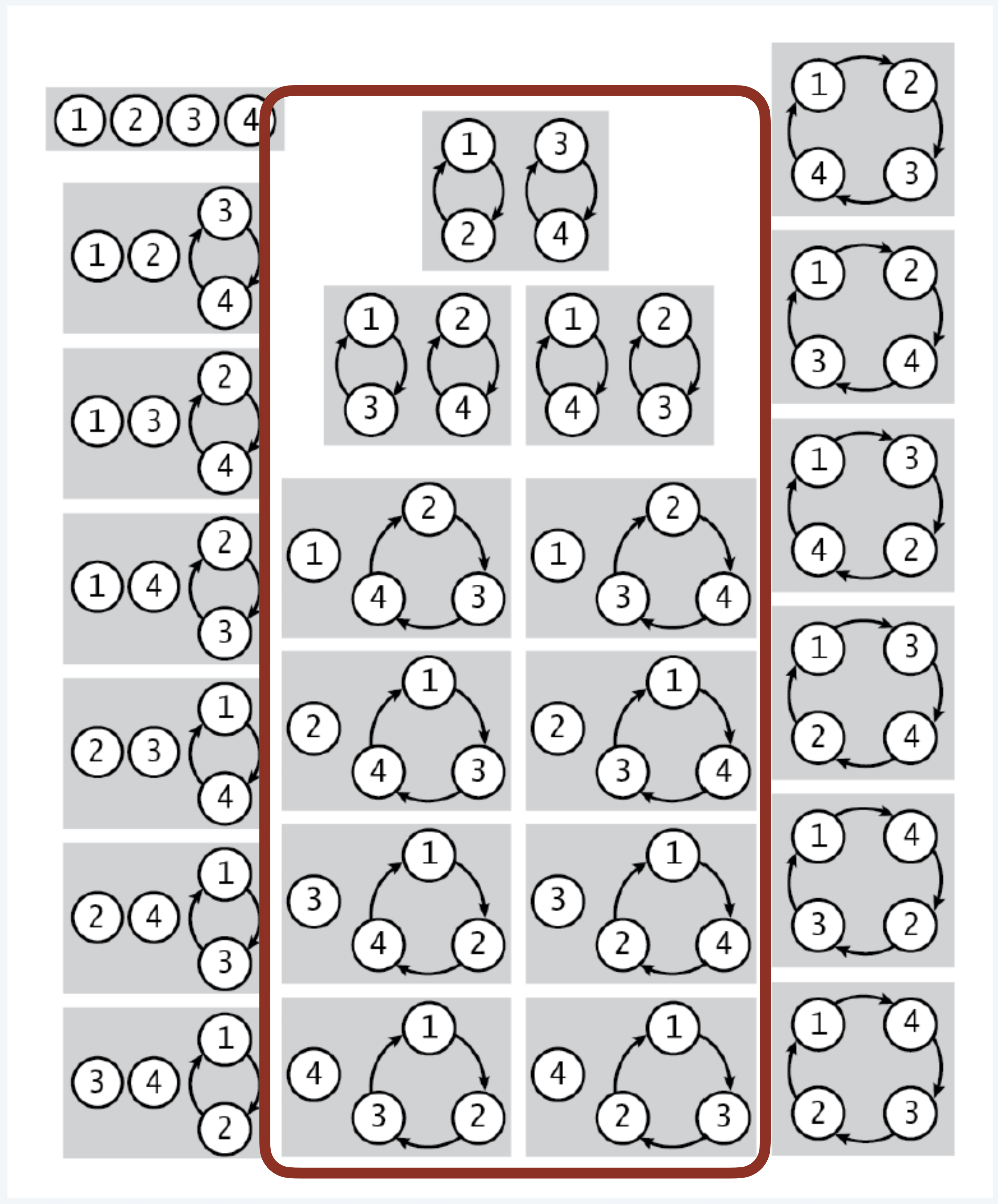

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AofA Chapter 7 Permutations (or analytic combinatorics) Q&A example

Q. What is the probability that a random perm of size *n* has exactly 2 cycles?

construction

$$P_2 = SET_2(CYC(Z))$$

EGF equation

$$P(z) = \frac{1}{2}\Big(\ln\big(\frac{1}{1-z}\big)\Big)^2$$

coefficient asymptotics

$$\frac{1}{2}\ln\big(\frac{1}{1-z}\big)^2 = \sum_{n\geq 0} p_n \frac{z^n}{n!}$$

$$\frac{1}{1-z}\ln\big(\frac{1}{1-z}\big) = \sum_{n\geq 1} p_n \frac{z^{n-1}}{(n-1)!}$$

technique: differentiate both sides

$$p_n = (n-1)!H_{n-1}$$

A. H_{n-1}/n

AofA Chapter 7 Permutations (or analytic combinatorics) Q&A example (improved)

Q. What is the probability that a random perm of size *n* has exactly 2 cycles?

construction

EGF equation

coefficient asymptotics

$$p_n = (n-1)!H_{n-1}$$

AofA Chapter 8 Strings and Tries Q&A example 1

Q. OGF for number of bitstrings not containing 01010 ?

constructions

$$E + (Z_0 + Z_1) \times B = B + P$$

$$Z_{01010} \times B = P + Z_{01} \times P + Z_{0101} \times P$$

GF equations

$$1 + 2zB(z) = B(z) + P(z)$$

$$z^5 B(z) = (1 + z^2 + z^4)P(z)$$

✗ NOT AN EXAM QUESTION
(too much calculation)

explicit form

$$B(z) = \frac{1 + z^2 + z^4}{z^5 + (1 - 2z)(1 + z^2 + z^4)}$$

But OK for a review problem set

AofA Chapter 8 Strings and Tries Q&A example 1 (improved version)

Q. Fill in the blanks in this OGF for the number of bitstrings not containing 01010.

constructions

$$E + (Z_0 + Z_1) \times B = B + P$$

$$Z_{01010} \times B = \boxed{}$$

GF equations

$$1 + 2zB(z) = \boxed{}$$

$$z^5 B(z) = (1 + z^2 + z^4)P(z)$$

explicit form

$$B(z) = \frac{1 + z^2 + z^4}{z^5 + (1 - 2z)(1 + z^2 + z^4)}$$

AofA Chapter 8 Strings and Tries Q&A example 2

Q. Rank these patterns by expected wait time in a random bit string.

00000	62
00001	32
01000	34
01010	36
10101	36

```
        0 1 0 0 0
        0 1 0 0 0    1
      0 1 0 0 0
    0 1 0 0 0
  0 1 0 0 0
0 1 0 0 0            z^4
```

$$c(z) = 1 + z^4$$

$$B_p(z) = \frac{c(z)}{z^P + (1-2z)c(z)}$$

wait time $2^P c(1/2)$

AofA Chapter 9 Words and Mappings Q&A example

Q. Find the probability that a random mapping has no singleton cycles.

constructions

$$C = Z \star SET(C) \qquad M = SET(CYC_{>1}(C))$$

EGF equations

$$C(z) = ze^{C(z)} \qquad M(z) = \exp\Big(\ln \frac{1}{1 - C(z)} - C(z)\Big) = \frac{e^{-C(z)}}{1 - C(z)}$$

coefficients via Lagrange inversion

✗ NOT AN EXAM QUESTION
(too much calculation)

Lagrange Inversion Theorem (Bürmann form).

If a GF $g(z) = \sum_{n \geq 1} g_n z^n$ satisfies the equation $z = f(g(z))$ with $f(0) = 0$ and $f'(0) \neq 0$ then, for any function $H(u)$,

$$[z^n]H(g(z)) = \frac{1}{n}[u^{n-1}]H'(u)\Big(\frac{u}{f(u)}\Big)^n$$

Still, you might want
this on your cheatsheet

asymptotic result

AofA Chapter 9 Words and Mappings Q&A example (improved)

Q. Give the EGF for *random mappings with no singleton cycles*.
Express your answer as a function of the Cayley function $C(z) = ze^{C(z)}$

constructions $C = Z \star SET(C)$ $\quad M = SET(CYC_{>1}(C))$

EGF equations $C(z) = ze^{C(z)}$

$$M(z) = \exp\Bigl(\ln \frac{1}{1 - C(z)} - C(z)\Bigr)$$

$$= \frac{e^{-C(z)}}{1 - C(z)}$$

Q. Find the probability that a *random mapping has no singleton cycles*.

A. Each entry can have any value but its own index, so the number of *N*-mappings with no singleton cycles is $(N-1)^N$

$$\frac{(N-1)^N}{N^N} = \Big(1 - \frac{1}{N}\Big)^N$$

$$\sim \frac{1}{e}$$

Mark your calendar

Wednesday **March 16**	**AofA review posted at 11:59AM**
Thursday **March 17**	**AofA review due at 11:59PM**
Friday **March 18**	**AC week 1 posted**
Wednesday **April 27**	**AC review posted at 11:59AM**
Tuesday **May 3**	**AC review due at 11:59PM**