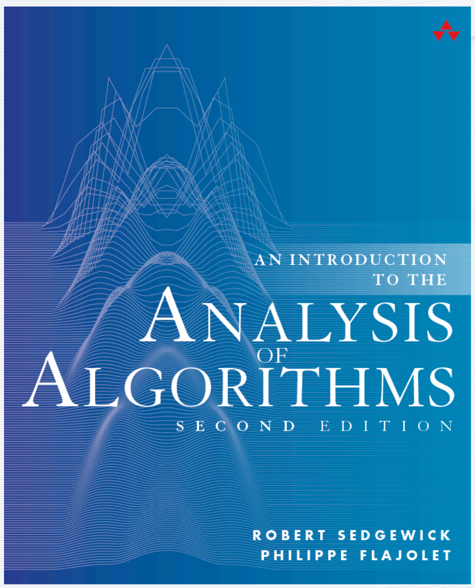
ANALYTIC COMBINATORICS
PART ONE



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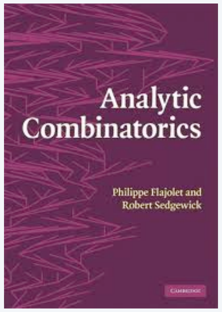
5. Analytic Combinatorics

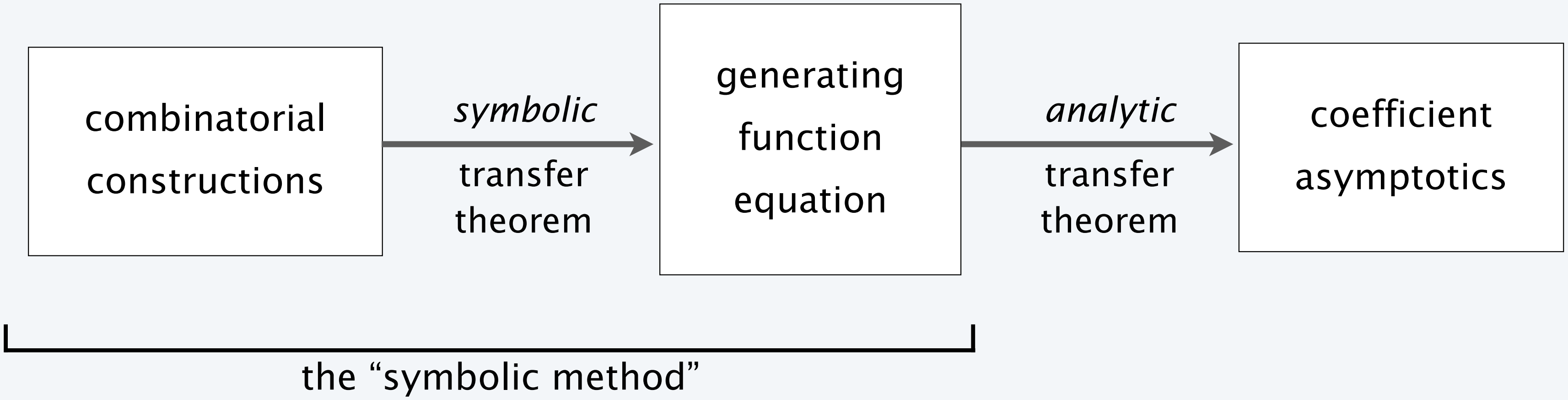
Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.

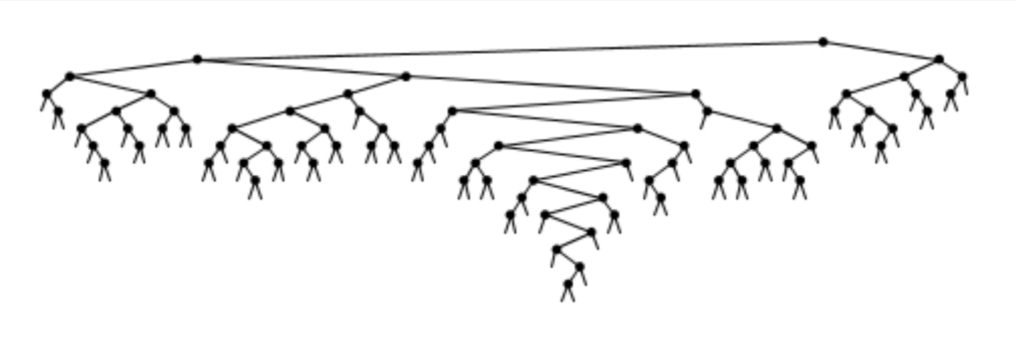




Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



T = E + Z × T × T

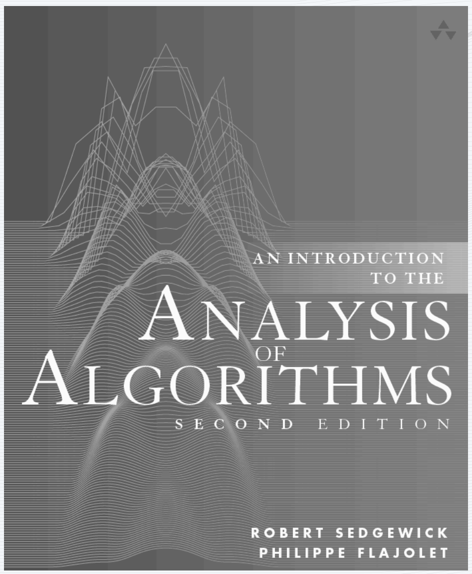
combinatorial construction

$$T(z) = 1 + zT(z)^2$$

GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient asymptotics



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5. Analytic Combinatorics

- **The symbolic method**
- Labelled objects
- Coefficient asymptotics
- Perspective

The symbolic method

is an approach for translating *combinatorial constructions* to *GF equations*

- Define a *class* of combinatorial objects.
- Define a notion of *size*.
- Define a *GF* whose coefficients count objects of the same size.
- Define *operations* suitable for constructive definitions of objects.
- Develop *translations* from constructions to operations on GFs.

Examples
A, B, Z
$\vert b \vert$
$A(z)$
$A \times B$
$A(z)B(z)$

Formal basis:

- A *combinatorial class* is a set of objects and a *size function*.
- An *atom* is an object of size 1.
- An *neutral object* is an atom of size 0.
- *A combinatorial construction* uses the union, product, and sequence operations to define a class in terms of atoms and other classes.

Building blocks

notation	*denotes*	*contains*
Z	atomic class	an atom
E	neutral class	neutral object
Φ	empty class	nothing

Unlabelled class example 1: natural numbers

Def. A *natural number* is a set (or a sequence) of atoms.

●	●●	●●●	●●●●	●●●●●
$I_1 = 1$	$I_2 = 1$	$I_3 = 1$	$I_4 = 1$	$I_5 = 1$

unary notation

counting sequence	*OGF*
$I_N = 1$	$\frac{1}{1-z}$

$$\sum_{N\geq 0} z^N = \frac{1}{1-z}$$

Unlabelled class example 2: bitstrings

Def. A *bitstring* is a sequence of 0 or 1 bits.

(empty)

$B_0 = 1$

0
1

$B_1 = 2$

00
01
10
11

$B_2 = 4$

000
001
010
011
100
101
110
111

$B_3 = 8$

0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

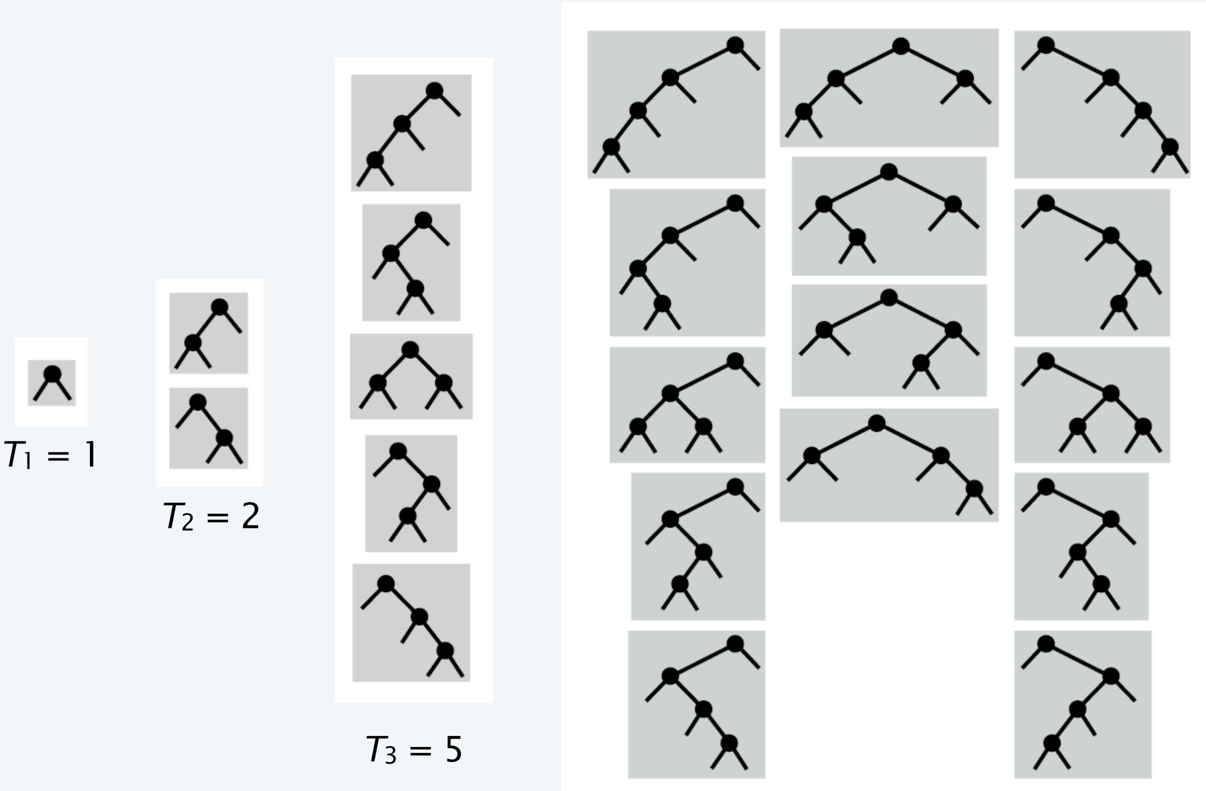
$B_4 = 16$

counting sequence	*OGF*
$B_N = 2^N$	$\dfrac{1}{1-2z}$

$$\sum_{N\geq 0} 2^N z^N = \sum_{N\geq 0} (2z)^N = \frac{1}{1-2z}$$

Unlabelled class example 3: binary trees

Def. A *binary tree* is empty or a sequence of a node and two binary trees



$T_1 = 1$ $T_2 = 2$ $T_3 = 5$ $T_4 = 14$

counting sequence	*OGF*
$T_N = \dfrac{1}{N+1}\dbinom{2N}{N}$	$\dfrac{1}{2z}(1-\sqrt{1-4z})$

Catalan numbers (see Lecture 3)

$$T(z) = 1 + zT(z)^2$$

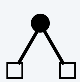
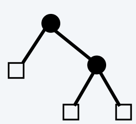
Combinatorial constructions for unlabelled classes

construction	*notation*	*semantics*
disjoint union	$A + B$	disjoint copies of objects from A and B
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B
sequence	$SEQ(A)$	sequences of objects from A

A and B are combinatorial classes of unlabelled objects

Ex 1. $(\,00 + 01\,) \times (\,101 + 110 + 111\,) = $ 00101 00110 00111 01101 01111

Ex 2. $\bullet \times SEQ(\bullet) = \bullet\ \ \bullet\bullet\ \ \bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\bullet\bullet\ \ \bullet\bullet\bullet\bullet\bullet\bullet\bullet\ \ \ldots$

Ex 3. $\square \times \bullet \times$  $=$ 

"unlabelled" ?? Stay tuned.

The symbolic method for unlabelled classes (transfer theorem)

Theorem. Let A and B be combinatorial classes of unlabelled objects with OGFs $A(z)$ and $B(z)$. Then

construction	*notation*	*semantics*	*OGF*
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ(A)$	sequences of objects from A	$\dfrac{1}{1 - A(z)}$

Proofs of transfers

are immediate from GF counting

A + B

$$\sum_{\gamma\in A+B} z^{|\gamma|} = \sum_{\alpha\in A} z^{|\alpha|} + \sum_{\beta\in B} z^{|\beta|} = A(z) + B(z)$$

A × B

$$\sum_{\gamma\in A\times B} z^{|\gamma|} = \sum_{\alpha\in A}\sum_{\beta\in B} z^{|\alpha|+|\beta|} = \Big(\sum_{\alpha\in A} z^{|\alpha|}\Big)\Big(\sum_{\beta\in B} z^{|\beta|}\Big) = A(z)B(z)$$

SEQ(A)

$$SEQ(A) \equiv \epsilon + A + A^2 + A^3 + A^4 + \ldots$$

$$1 + A(z) + A(z)^2 + A(z)^3 + A(z)^4 + \ldots = \frac{1}{1 - A(z)}$$

Symbolic method: binary trees

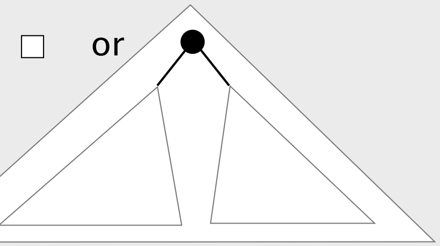
How many binary trees with N nodes?

Class	T, the class of all binary trees
Size	$\lvert t \rvert$, the number of internal nodes in t
OGF	$T(z) = \sum_{t \in T} z^{\lvert t \rvert} = \sum_{N \geq 0} T_N z^N$

Atoms

type	*class*	*size*	*GF*
external node	$Z_\square$	0	1
internal node	$Z_\bullet$	1	z

“a binary tree is an external node or an internal node connected to two binary trees”



Construction $\quad T = Z_\square + T \times Z_\bullet \times T$

OGF equation $\quad T(z) = 1 + zT(z)^2$

$$[z^N]T(z) = \frac{1}{N+1}\binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N^3}}$$

see Lecture 3 and stay tuned.

Symbolic method: binary trees

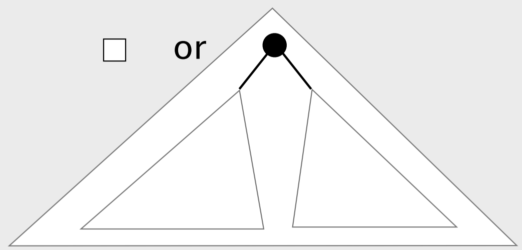
How many binary trees with N *external* nodes?

Class	T, the class of all binary trees
Size	$\boxed{t}$, the number of *external* nodes in t
OGF	$T^{\Box}(z) = \sum_{t\in T} z^{\boxed{t}}$

Atoms

type	*class*	*size*	*GF*
external node	$Z_{\Box}$	1	z
internal node	$Z_{\bullet}$	0	1

"a binary tree is an external node or an internal node connected to two binary trees"



Construction $$T = Z_{\Box} + T \times Z_{\bullet} \times T$$

OGF equation $$T^{\Box}(z) = z + T^{\Box}(z)^2$$

$$T^{\Box}(z) = zT(z)$$

$$[z^N]T^{\Box}(z) = [z^{N-1}]T(z) = \frac{1}{N}\binom{2N-2}{N-1}$$

same as # binary trees with N–1 internal nodes

Symbolic method: binary strings

Warmup: How many binary strings with *N* bits?

Class	*B*, the class of all binary strings
Size	$\|b\|$, the number of bits in *b*
OGF	$B(z) = \sum_{b \in B} z^{\|b\|} = \sum_{N \geq 0} B_N z^N$

Atoms

type	*class*	*size*	*GF*
0 bit	Z_0	1	*z*
1 bit	Z_1	1	z

Construction $$B = SEQ(Z_0 + Z_1)$$

"a binary string is a sequence of 0 bits and 1 bits"

OGF equation $$B(z) = \frac{1}{1-2z}$$

$$[z^N]B(z) = 2^N \quad \checkmark$$

Symbolic method: binary strings (alternate)

Warmup: How many binary strings with N bits?

Class	B, the class of all binary strings
Size	$\|b\|$, the number of bits in b
OGF	$B(z) = \sum_{b \in B} z^{\|b\|} = \sum_{N \geq 0} B_N z^N$

Atoms

type	*class*	*size*	*GF*
0 bit	Z_0	1	z
1 bit	Z_1	1	z

Construction $$B = E + (Z_0 + Z_1) \times B$$

"a binary string is empty or a bit followed by a binary string"

OGF equation $$B(z) = 1 + 2zB(z)$$

Solution $$B(z) = \frac{1}{1 - 2z}$$

$$[z^N]B(z) = 2^N \quad ✓$$

Symbolic method: binary strings with restrictions

Ex. How many N-bit binary strings have no two consecutive 0s?

$T_0 = 1$	$T_1 = 2$	$T_2 = 3$	$T_3 = 5$	$T_4 = 8$	$T_5 = 13$
	0	01	011	0101	01011
	1	10	010	0110	01010
		11	101	0111	01101
			110	1011	01110
			111	1010	01111
				1101	10101
				1110	10110
				1111	10111
					11011
					11010
					11101
					11110
					11111

Stay tuned for general treatment (Chapter 8)

Symbolic method: binary strings with restrictions

Ex. How many N-bit binary strings have no two consecutive 0s?

Class	B_{00}, the class of binary strings with no 00
Size	$\|b\|$, the number of bits in b
OGF	$B_{00}(z) = \sum_{b \in B_{00}} z^{\|b\|}$

Atoms

type	*class*	*size*	*GF*
0 bit	Z_0	1	z
1 bit	Z_1	1	z

Construction $$B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$$

"a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00"

OGF equation $$B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$$

solution $$B_{00}(z) = \frac{1+z}{1-z-z^2}$$

$$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$$

1, 2, 5, 8, 13, ... ✓

3,

Symbolic method: many, many examples to follow

How many ... with ... ?

Class	
Size	
OGF	

Atoms

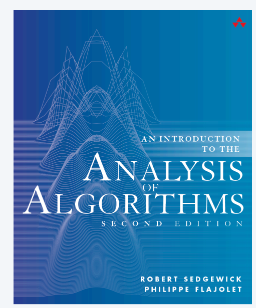
type	*class*	*size*	*GF*

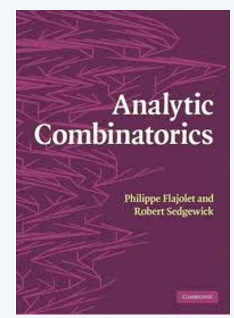
Construction

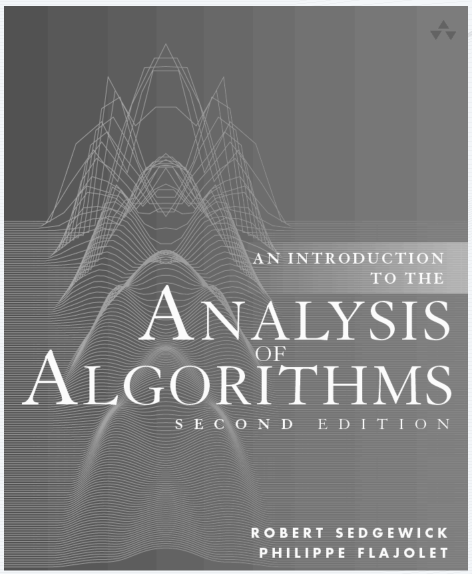
"a ... is either ... or ... and ..."

OGF equation

solution







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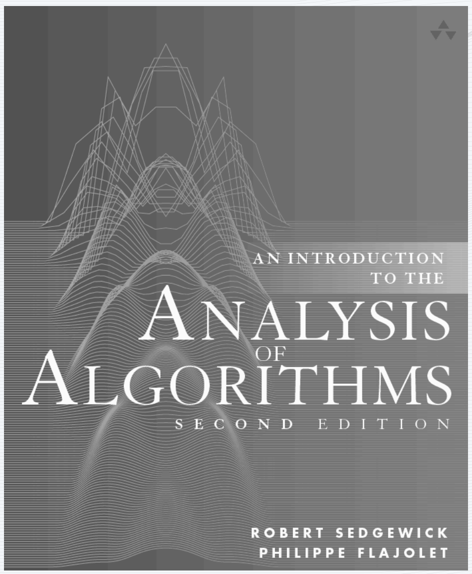
5. Analytic Combinatorics

- **The symbolic method**
- Labelled objects
- Coefficient asymptotics
- Perspective



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5. Analytic Combinatorics

- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
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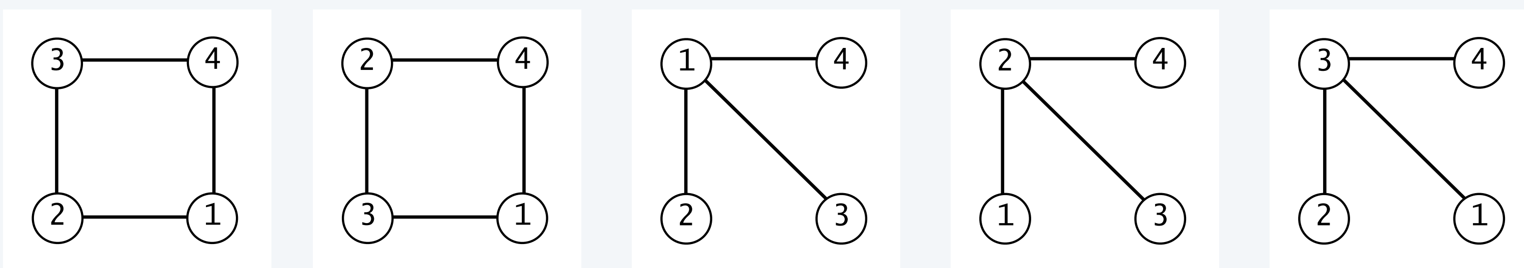
Labelled combinatorial classes

have objects composed of N atoms, labelled with the integers 1 through N.

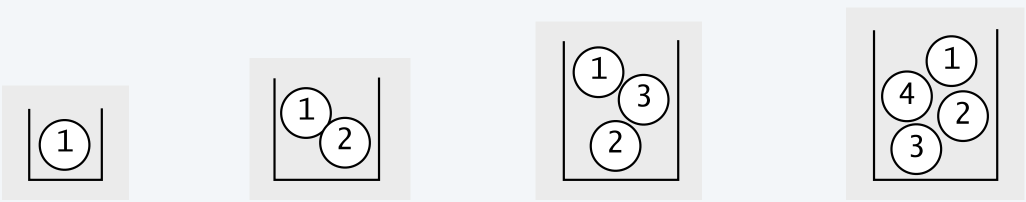
Ex. Different unlabelled objects

Ex. Different labelled objects



Labelled class example 1: urns

Def. An *urn* is a set of labelled atoms.



$U_1 = 1$ $U_2 = 1$ $U_3 = 1$ $U_4 = 1$

counting sequence	*EGF*
$U_N = 1$	e^z

$$\sum_{N \geq 0} \frac{z^N}{N!} = e^z$$

Labelled class example 2: permutations

Def. A *permutation* is a sequence of labelled atoms.

1

$P_1 = 1$

1 2
2 1

$P_2 = 1$

1 2 3
2 1 3
3 1 2
1 3 2
2 3 1
3 2 1

$P_3 = 2$

1 2 3 4   1 2 4 3
2 1 3 4   2 1 4 3
3 1 2 4   3 1 4 2
4 1 2 3   4 1 3 2
1 3 2 4   1 3 4 2
2 3 1 4   2 3 4 1
3 2 1 4   3 2 4 1
4 2 1 3   4 2 3 1
1 4 2 3   1 4 3 2
2 4 1 3   2 4 3 1
3 4 1 2   3 4 2 1
4 3 1 2   4 3 2 1

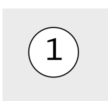
$P_4 = 6$

counting sequence	*EGF*
$P_N = N!$	$\dfrac{1}{1-z}$

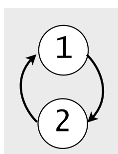
$$\sum_{N\geq 0}\frac{N!z^N}{N!} = \sum_{N\geq 0} z^N = \frac{1}{1-z}$$

Labelled class example 3: cycles

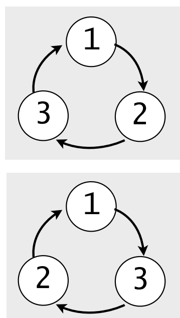
Def. A *cycle* is a cyclic sequence of labelled atoms



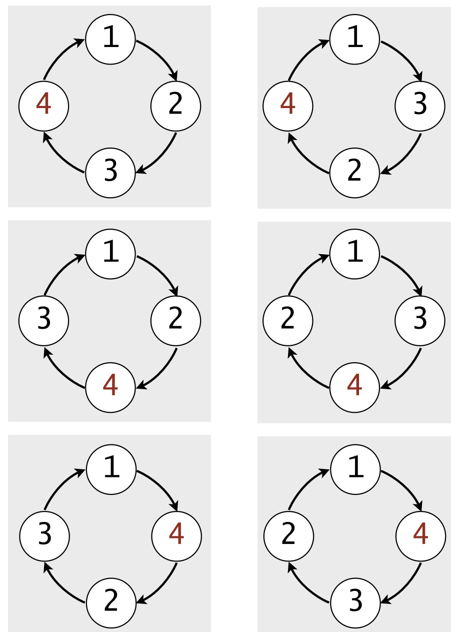
$C_1 = 1$



$C_2 = 1$



$C_3 = 2$



$C_4 = 6$

counting sequence	*EGF*
$C_N = (N-1)!$	$\ln \frac{1}{1-z}$

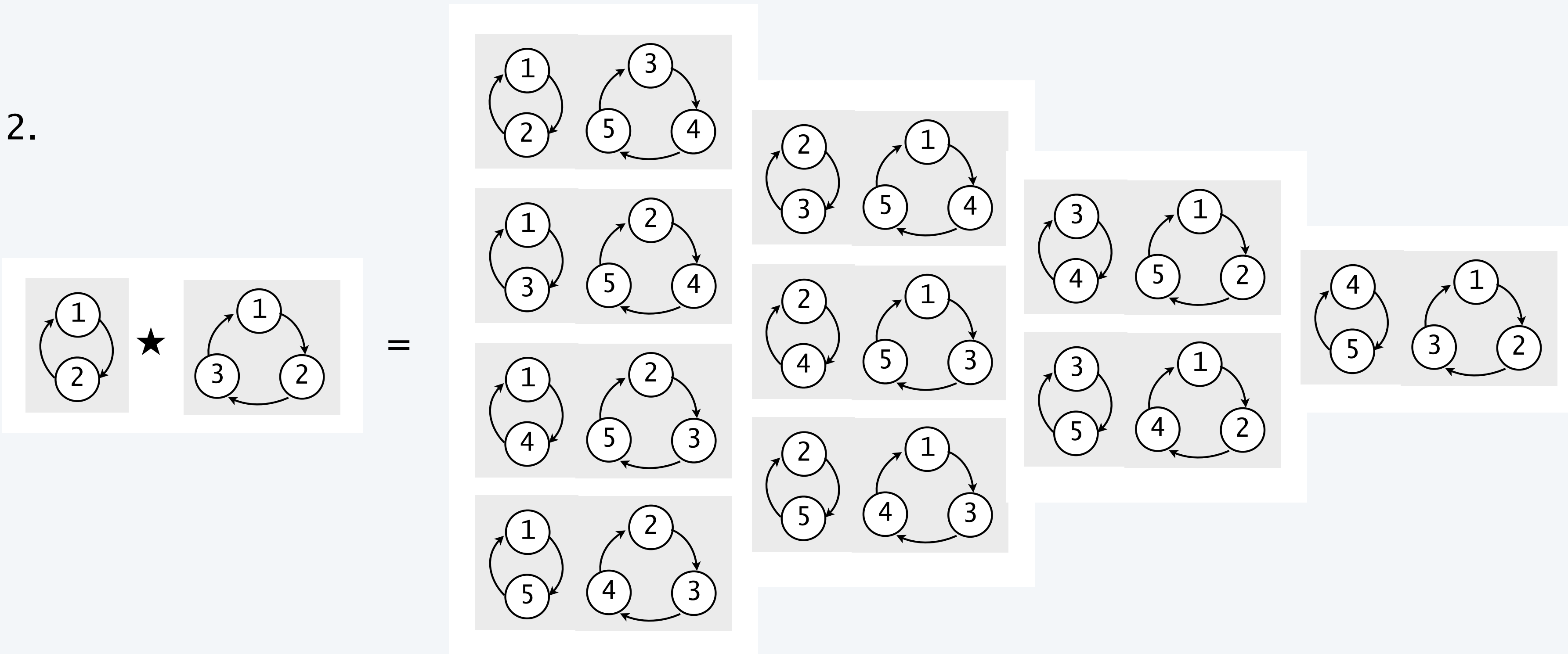
$$\sum_{N\geq 1} \frac{(N-1)!z^N}{N!} = \sum_{N\geq 1} \frac{z^N}{N} = \ln \frac{1}{1-z}$$

Star product operation

Analog to Cartesian product requires *relabelling in all consistent ways.*

Ex 1. 1 ★ 1 2 3 = 1 2 3 4 2 1 3 4 3 1 2 4 4 1 2 3

Ex 2.



Combinatorial constructions for labelled classes

construction	*notation*	*semantics*
disjoint union	$A + B$	disjoint copies of objects from A and B
labelled product	$A \star B$	ordered pairs of copies of objects, one from A and one from B
sequence	$SEQ(A)$	sequences of objects from A
set	$SET(A)$	sets of objects from A
cycle	$CYC(A)$	cyclic sequences of objects from A

A and B are combinatorial classes of labelled objects

The symbolic method for labelled classes (transfer theorem)

Theorem. Let A and B be combinatorial classes of labelled objects with EGFs $A(z)$ and $B(z)$. Then

construction	*notation*	*semantics*	*EGF*
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
labelled product	$A \star B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ_k(A)$	k- sequences of objects from A	$A(z)^k$
	$SEQ(A)$	sequences of objects from A	$\frac{1}{1-A(z)}$
set	$SET_k(A)$	k-sets of objects from A	$A(z)^k/k!$
	$SET(A)$	sets of objects from A	$e^{A(z)}$
cycle	$CYC_k(A)$	k-cycles of objects from A	$A(z)^k/k$
	$CYC(A)$	cycles of objects from A	$\ln \frac{1}{1-A(z)}$

The symbolic method for labelled classes: basic constructions

class	*construction*	*EGF*	*counting sequence*
urns	$U = SET(Z)$	$U(z) = e^z$	$U_N = 1$
cycles	$C = CYC(Z)$	$C(z) = \ln \frac{1}{1-z}$	$C_N = (N-1)!$
permutations	$P = SEQ(Z)$	$P(z) = \frac{1}{1-z}$	$P_N = N!$
	$P = E + Z \star P$		

construction	*notation*	*EGF*
disjoint union	$A + B$	$A(z) + B(z)$
labelled product	$A \star B$	$A(z)B(z)$
sequence	$SEQ_k (A)$	$A(z)^k$
	$SEQ (A)$	$\frac{1}{1-A(z)}$
set	$SET_k (A)$	$A(z)^k/k!$
	$SET (A)$	$e^{A(z)}$
cycle	$CYC_k (A)$	$A(z)^k/k$
	$CYC (A)$	$\ln \frac{1}{1-A(z)}$

Proofs of transfers

are immediate from GF counting

$A + B$

$$\sum_{\gamma\in A+B}\frac{z^{|\gamma|}}{|\gamma|!}=\sum_{\alpha\in A}\frac{z^{|\alpha|}}{|\alpha|!}+\sum_{\beta\in B}\frac{z^{|\beta|}}{|\beta|!}=A(z)+B(z)$$

$A \star B$

$$\sum_{\gamma\in\mathcal{A}\times\mathcal{B}}\frac{z^{|\gamma|}}{|\gamma|!}=\sum_{\alpha\in\mathcal{A}}\sum_{\beta\in\mathcal{B}}\binom{|\alpha|+|\beta|}{|\alpha|}\frac{z^{|\alpha|+|\beta|}}{(|\alpha|+|\beta|)!}=\Big(\sum_{\alpha\in\mathcal{A}}\frac{z^{|\alpha|}}{|\alpha|!}\Big)\Big(\sum_{\beta\in\mathcal{B}}\frac{z^{|\beta|}}{|\beta|!}\Big)=A(z)B(z)$$

Notation. We write A^2 for $A \star A$, A^3 for $A \star A \star A$, etc.

Proofs of transfers

are immediate from GF counting

$$A(z)^k = \sum_{N\geq 0}\{\#k\text{-sequences of size } N\}\frac{z^N}{N!} = \sum_{N\geq 0} k\{\#k\text{-cycles of size } N\}\frac{z^N}{N!} = \sum_{N\geq 0} k!\{\#k\text{-sets of size } N\}\frac{z^N}{N!}$$

$$\frac{A(z)^k}{k} = \sum_{N\geq 0}\{\#k\text{-cycles of size } N\}\frac{z^N}{N!}$$

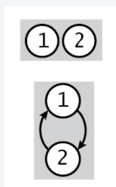
$$\frac{A(z)^k}{k!} = \sum_{N\geq 0}\{\#k\text{-sets of size } N\}\frac{z^N}{N!}$$

class	*construction*	*EGF*
k-sequence	$SEQ_k(A)$	$A(z)^k$
sequence	$SEQ_k(A) = SEQ_0(A) + SEQ_1(A) + SEQ_2(A) + \ldots$	$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \dfrac{1}{1-A(z)}$
k-cycle	$CYC_k(A)$	$\dfrac{A(z)^k}{k}$
cycle	$CYC_k(A) = CYC_0(A) + CYC_1(A) + CYC_2(A) + \ldots$	$1 + \dfrac{A(z)}{1} + \dfrac{A(z)^2}{2} + \dfrac{A(z)^3}{3} + \ldots = \ln\dfrac{1}{1-A(z)}$
k-set	$SET_k(A)$	$\dfrac{A(z)^k}{k!}$
set	$SET_k(A) = SET_0(A) + SET_1(A) + SET_2(A) + \ldots$	$1 + \dfrac{A(z)}{1!} + \dfrac{A(z)^2}{2!} + \dfrac{A(z)^3}{3!} + \ldots = e^{A(z)}$

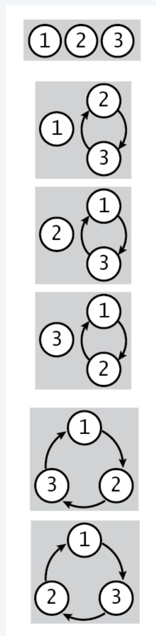
Labelled class example 4: sets of cycles

Q. How many sets of cycles of labelled atoms?

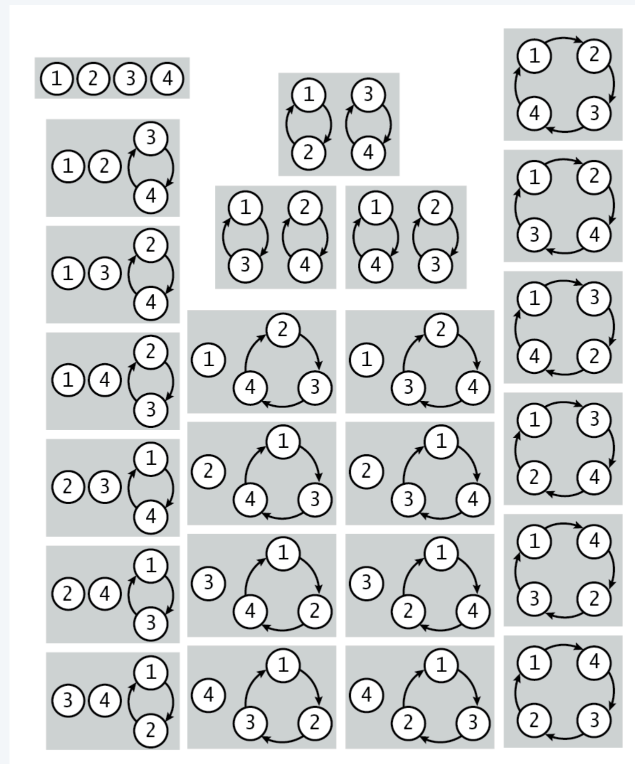
$P^*_1 = 1$



$P^*_2 = 2$



$P^*_3 = 6$



$P^*_4 = 24$

Symbolic method: sets of cycles

How many sets of cycles of length N ?

Class	P^*, the class of all sets of cycles of atoms
Size	$\lvert p\rvert$, the number of atoms in p
EGF	$P^*(z) = \sum_{p \in P^*} \frac{z^{\lvert p\rvert}}{\lvert p\rvert!} = \sum_{N \geq 0} P_N^* \frac{z^N}{N!}$

Atom

type	*class*	*size*	*GF*
labelled atom	Z	1	z

Construction $P^* = SET(CYC(Z))$

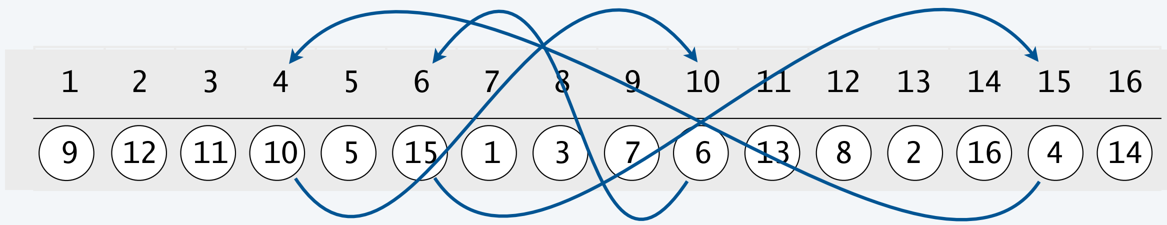
OGF equation $P^*(z) = \exp\left(\ln \frac{1}{1-z}\right) = \frac{1}{1-z}$

Counting sequence $P_N^* = N![z^N]P^*(z) = N!$

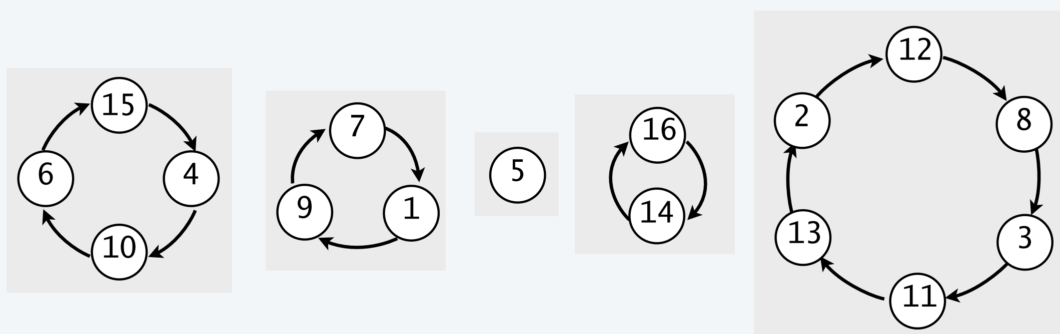
Aside: A combinatorial bijection

A permutation is a set of cycles.

Standard representation



Set of cycles representation



Derangements

N people go to the opera and leave their hats on a shelf in the cloakroom. When leaving, they each grab a hat at random.

Q. *What is the probability that nobody gets their own hat ?*

Definition. A derangement is a permutation with no singleton cycles

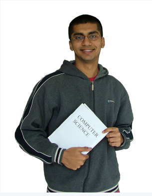
Derangements (various versions)

A group of N people go to the opera and leave their hats in the cloakroom. When leaving, they each grab a hat at random.

Q. *What is the probability that nobody gets their own hat*?

A professor returns exams to N students by passing them out at random.

Q. *What is the probability that nobody gets their own exam*?

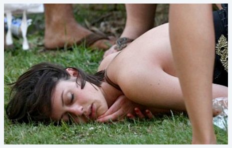


A group of N sailors go ashore for revelry that leads to a state of inebriation. When returning, they each end up sleeping in a random cabin.

Q. *What is the probability that nobody sleeps in their own cabin*?

A group of N students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

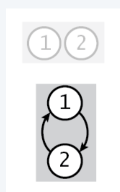
Q. *What is the probability that nobody ends up in their own room*?



Derangements

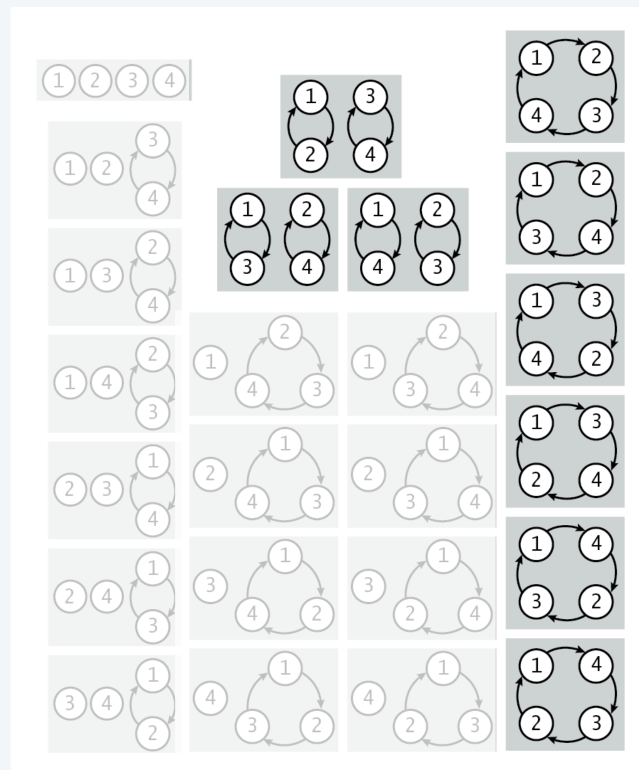
are permutations with no singleton cycles.

$D_1 = 0$



$D_2 = 1$

$D_3 = 2$



$D_4 = 9$

Symbolic method: derangements

How many derangements of length N?

Class	D, the class of all derangements
Size	$\lvert p \rvert$, the number of atoms in p
EGF	$D(z) = \sum_{d \in D} \frac{z^{\lvert d \rvert}}{\lvert d \rvert !} = \sum_{N \geq 0} D_N \frac{z^N}{N!}$

Atom

type	*class*	*size*	*GF*
labelled atom	Z	1	z

Construction $D = SET(CYC_{>1}(Z))$

"Derangements are permutations with no singleton cycles"

OGF equation $$D(z) = e^{z^2/2+z^3/3+z^4/4+\ldots} = \exp\Big(\ln \frac{1}{1-z} - z\Big) = \frac{e^{-z}}{1-z}$$

Expansion $$[z^N]D(z) \equiv \frac{D_N}{N!} = \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} \sim \frac{1}{e}$$

probability that a random permutation is a derangement

simple convolution

see "Asymptotics" lecture

Alternate derivation

$Set(Z) \star D = P$

$$e^z D(z) = \frac{1}{1-z}$$

Derangements

A group of *N* students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. *What is the probability that nobody ends up in their own room* ?

A. $\frac{1}{e} \doteq 0.36788$

Derangements

A group of N graduating seniors each throw their hats in the air and each catch a random hat.

Q. *What is the probability that nobody gets their own hat back* ?

A. $\frac{1}{e} \doteq 0.36788$

Generalized derangements

In the hats-in-the-air scenario, a student can get her hat back by "following the cycle".

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	12	11	10	5	15	1	3	7	6	13	8	2	16	4	14

Q. *What is the probability that all cycles are of length > M ?*

Symbolic method: generalized derangements

How many permutations of length N have no cycles of length $\leq M$?

Class	D_M, the class of all generalized derangements
Size	$\lvert d \rvert$, the number of atoms in d
EGF	$D_M(z) = \sum_{d \in D_M} \frac{z^{\lvert d \rvert}}{\lvert d \rvert !} = \sum_{N \geq 0} D_{MN} \frac{z^N}{N!}$

Atom

type	*class*	*size*	*GF*
labelled atom	Z	1	z

Construction $\quad D_M = SET(CYC_{>M}(Z))$

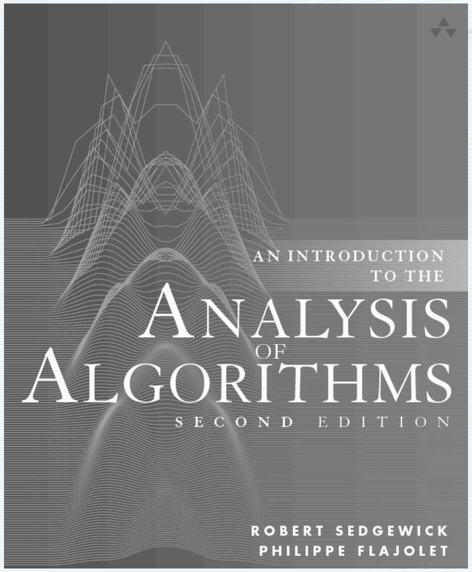
OGF equation

$$D_M(z) = e^{\frac{z^{M+1}}{M+1} + \frac{z^{M+2}}{M+2} + \cdots} = \exp\left(\ln \frac{1}{1-z} - z - z^2/2 - \ldots - z^M/M\right)$$

$$= \frac{e^{-z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots \frac{z^M}{M}}}{1-z}$$

Expansion $\quad D_{MN} =$?? M-way convolution (stay tuned)

ANALYTIC COMBINATORICS
PART ONE



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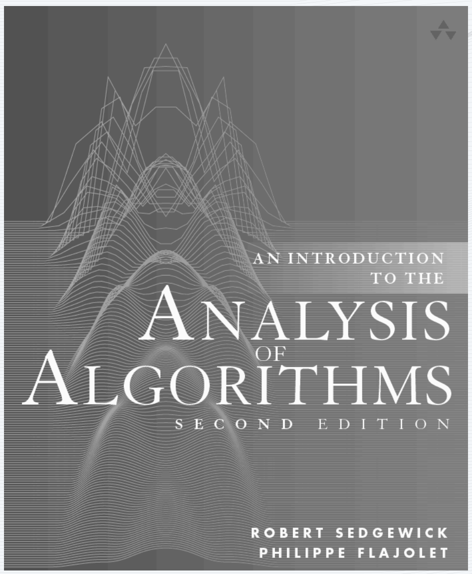
5. Analytic Combinatorics

- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
- Perspective



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5. Analytic Combinatorics

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Generating coefficient asymptotics

are often *immediately* derived via general "analytic" transfer theorems.

Example 1. Taylor's theorem

Theorem. If $f(z)$ has N derivatives, then $[z^N]f(z) = f^{(N)}(0)/N!$

Example 2. Rational functions transfer theorem (see "Asymptotics" lecture)

Theorem. If $f(z)$ and $g(z)$ are polynomials, then

$$[z^n]\frac{f(z)}{g(z)} = -\frac{\beta f(1/\beta)}{g'(\beta)}\beta^n$$

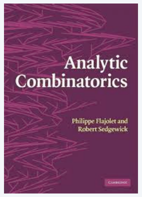
where $1/\beta$ is the largest root of g (provided that it has multiplicity 1).

see "Asymptotics" lecture for general case

Example 3. Radius-of-convergence transfer theorem

[see next slide]

Most are based on complex asymptotics. Stay tuned for Part 2



Radius-of-convergence transfer theorem

Theorem. If $f(z)$ has radius of convergence >1 with $f(1) \neq 0$, then

$$[z^n]\frac{f(z)}{(1-z)^\alpha} \sim f(1)\binom{n+\alpha-1}{n} \sim \frac{f(1)}{\Gamma(\alpha)}n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \ldots$

convolution,
$f_1 + f_2 + \ldots + f_n \sim f(1)$

standard asymptotics
with generalized
binomial coefficient

Gamma function
(generalized factorial)

$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$
$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$
$\Gamma(N+1) = N!$
$\Gamma(1) = 1$
$\Gamma(1/2) = \sqrt{\pi}$

Corollary. If $f(z)$ has radius of convergence >ρ with $f(\rho) \neq 0$, then

$$[z^n]\frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \ldots$

Radius-of-convergence transfer theorem: applications

Corollary. If $f(z)$ has radius of convergence $>\rho$ with $f(\rho) \neq 0$, then

$$[z^n]\frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{n}n^{\alpha-1}$$

$-n$

for any real $\alpha \notin 0, -1, -2, \ldots$

Ex 1: Catalan

$$T(z) = \frac{1}{2z}(1-\sqrt{1-4z})$$

$$[z^N]T(z) \sim \frac{4^N}{\sqrt{\pi N^3}}$$

$\rho = 1/4 \qquad \alpha = -1/2 \qquad f(z) = -1/2$

$\Gamma(-1/2) = -2\Gamma(1/2) = -2\sqrt{\pi}$

Ex 2: Derangements

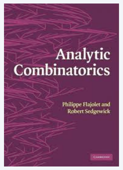
$$D_M(z) = \frac{e^{-z-z^2/2\ldots-z^M/M}}{1-z}$$

$$[z^N]D_M(z) \sim \frac{N!}{e^{H_M}}$$

$\rho = 1 \quad \alpha = 1 \quad f(z) = e^{-z-z^2/2\ldots-z^M/M}$

Transfer theorems based on complex asymptotics

provide *universal laws* of sweeping generality



Example: Context-free constructions

A *system* of combinatorial constructions

$$
\begin{aligned}
< \mathbf{G}_0 > &= OP_0(< \mathbf{G}_0 >, < \mathbf{G}_1 >, \ldots, < \mathbf{G}_\mathbf{t} >)\\
< \mathbf{G}_1 > &= OP_1(< \mathbf{G}_0 >, < \mathbf{G}_1 >, \ldots, < \mathbf{G}_\mathbf{t} >)\\
&\ldots\\
< \mathbf{G}_\mathbf{t} > &= OP_t(< \mathbf{G}_0 >, < \mathbf{G}_1 >, \ldots, < \mathbf{G}_\mathbf{t} >)
\end{aligned}
$$

symbolic method

transfers to a system of GF equations

$$
\begin{aligned}
G_0(z) &= F_0(G_0(z), G_1(z), \ldots, G_t(z))\\
G_1(z) &= F_1(G_0(z), G_1(z), \ldots, G_t(z))\\
&\ldots\\
G_t(z) &= F_t(G_0(z), G_1(z), \ldots, G_t(z))
\end{aligned}
$$

Grobner basis elimination

that reduces to a *single* GF equation

$$G_0(z) = F(G_0(z), G_1(z), \ldots, G_t(z))$$

Drmota-Lalley-Woods theorem

that has an *explicit* solution

$$G(z) \sim c - a\sqrt{1 - bz}$$

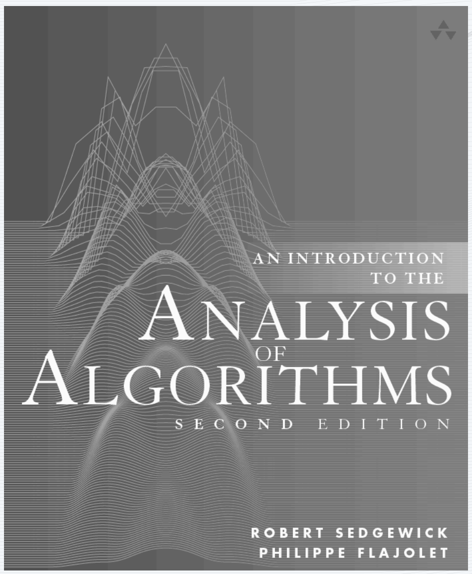
singularity analysis

that transfers to a *simple* asymptotic form

$$G_N \sim \frac{a}{2\sqrt{\pi N^3}} b^N$$

!!

Stay tuned for many more (in Part 2).



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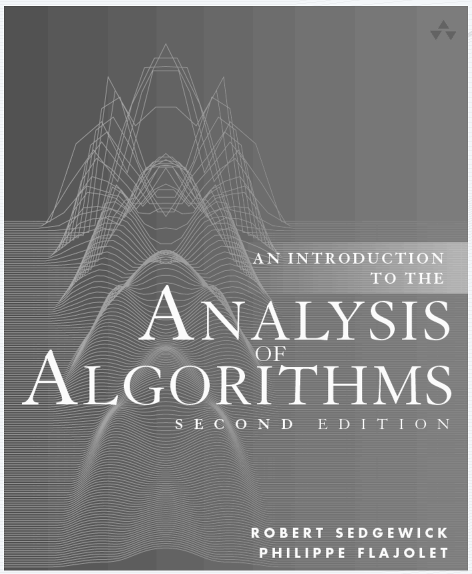
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- **Coefficient asymptotics**
- Perspective



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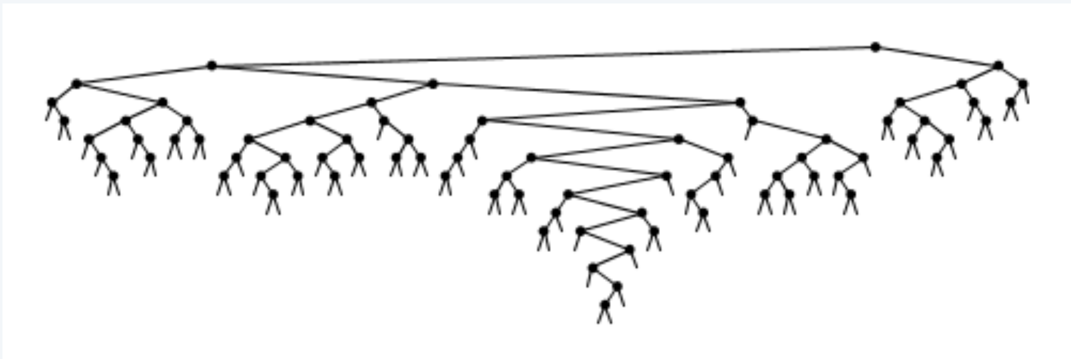
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- **Perspective**

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T \;\longrightarrow\; T(z) = \frac{1}{2z}(1 - \sqrt{1-4z}) \;\longrightarrow\; T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

combinatorial construction

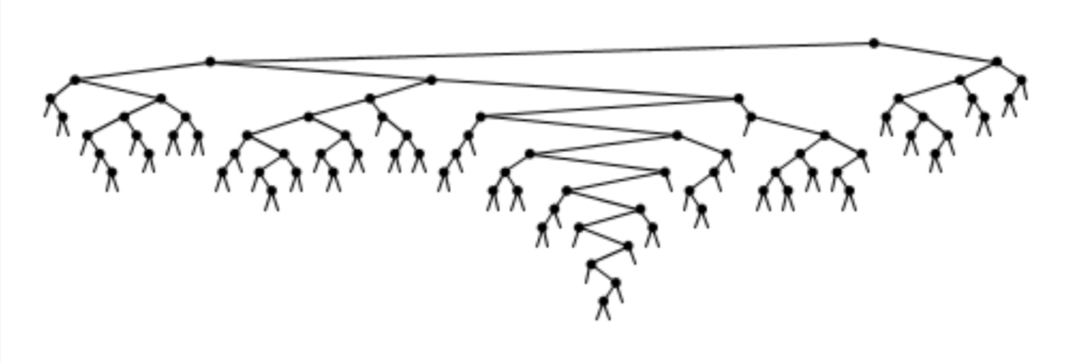
GF

coefficient asymptotics

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$\text{T} = \text{E} + \text{Z} \times \text{T} \times \text{T}$$

combinatorial construction

$$T(z) = 1 + zT(z)^2$$

GF equation

Note: With complex asymptotics, we can transfer directly from GF equation (no need to solve it). See Part 2.

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient asymptotics

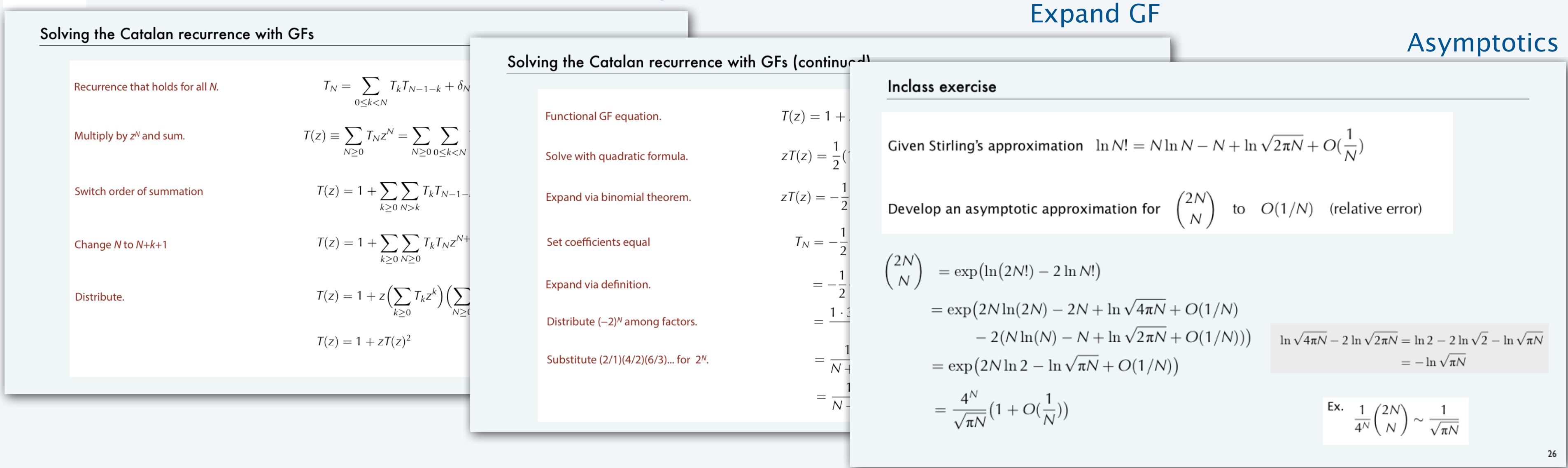
Old vs. New: Two ways to count binary trees

Old

Recurrence → GF

Expand GF

Asymptotics



New

$$\mathrm{T} = \mathrm{E} + \mathrm{Z} \times \mathrm{T} \times \mathrm{T} \longrightarrow T(z) = 1 + zT(z)^2 \longrightarrow T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many generalized derangements?

$$D_M = SET(CYC_{>M}(Z)) \longrightarrow \frac{e^{-z-z^2/2\ldots-z^M/M}}{1-z} \longrightarrow \sim \frac{N!}{e^{H_M}}$$

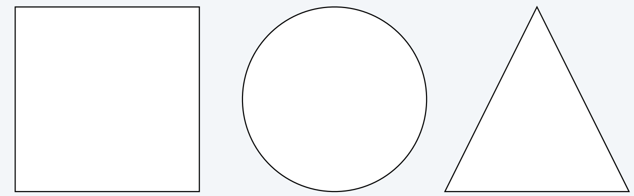
combinatorial construction

GF equation

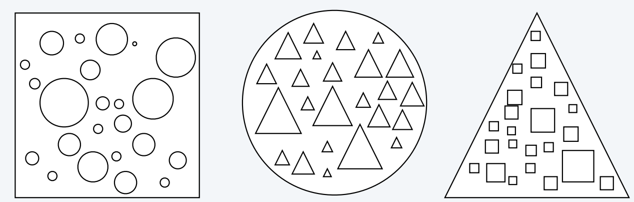
coefficient asymptotics

A standard paradigm for analytic combinatorics

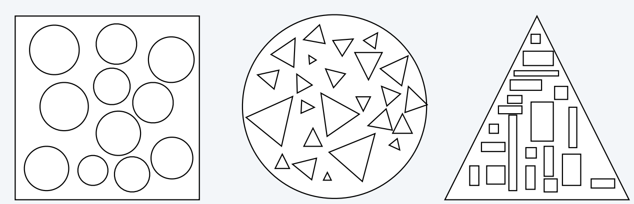
Fundamental constructs
- elementary or trivial
- confirm intuition



Compound constructs
- many possibilities
- classical combinatorial objects
- expose underlying structure

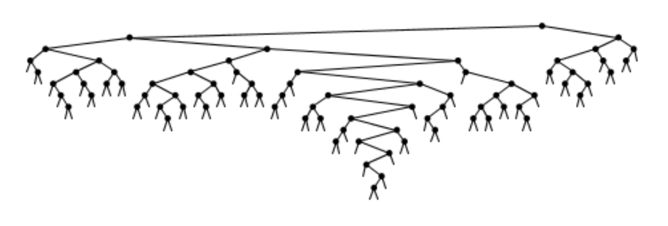


Variations
- unlimited possibilities
- *not* easily analyzed otherwise



Combinatorial parameters

are handled as two counting problems via cumulated costs.



Ex: How many leaves in a random binary tree?

1. Count trees

$$T = E + Z \times T \times T \longrightarrow T(z) = \frac{1}{2z}(1 - \sqrt{1-4z}) \longrightarrow T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

2. Count leaves in all trees

$$T = E + Z \times T \times T \longrightarrow T_u(1, z) = \frac{z}{\sqrt{1-4z}} \longrightarrow C_N \sim \frac{4^{N-1}}{\sqrt{\pi N}}$$

Symbolic method works for BGFs (see text)

3. Divide

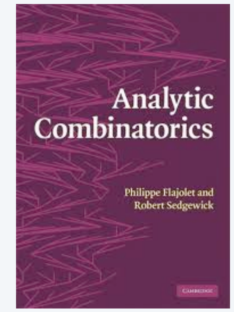
$$\frac{C_N}{T_N} \sim \frac{N}{4}$$

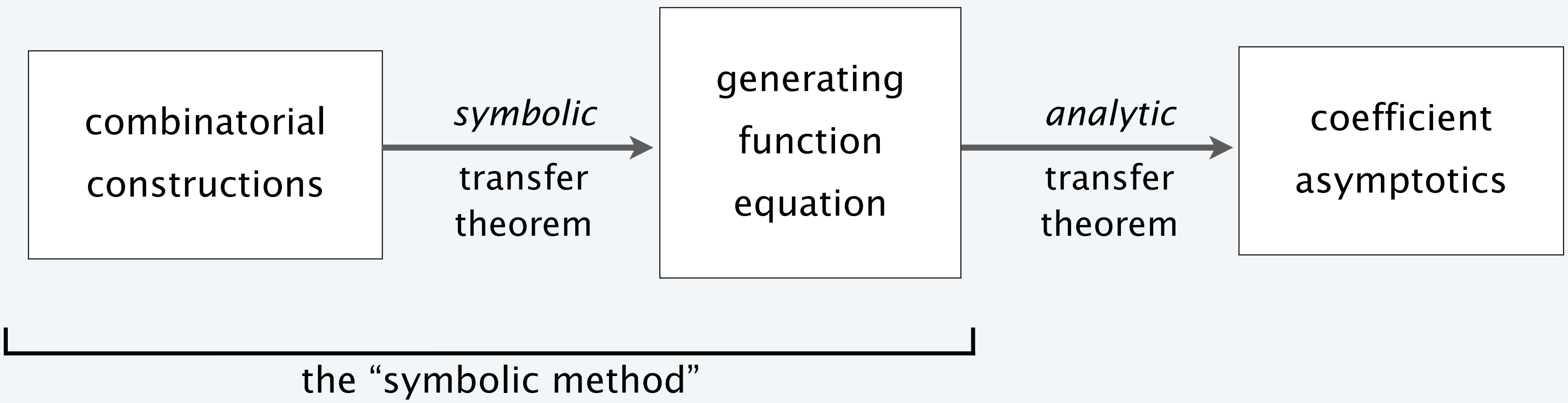
Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.

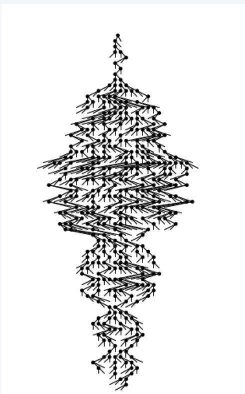


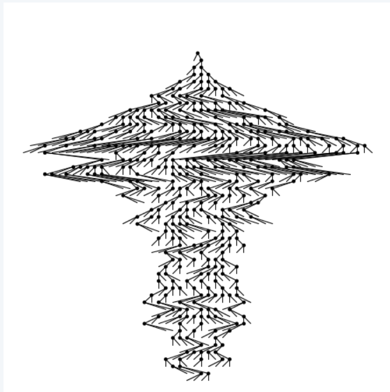


Stay tuned

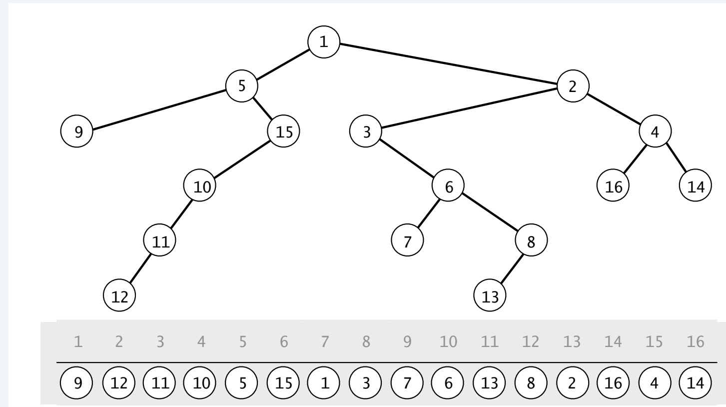
for many applications of analytic combinatorics *and applications to the analysis of algorithms*

Trees

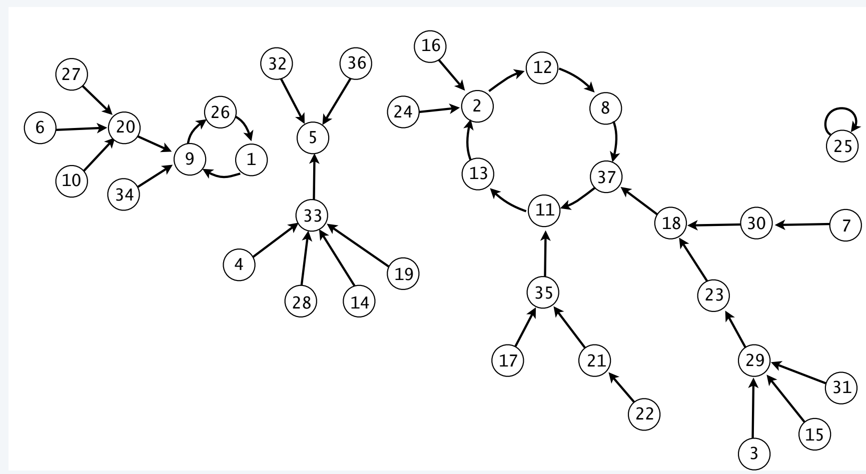




Permutations



Mappings



Bitstrings

1011111010010100110011**1000**1001111101101101000001111000011001110111011111101011000

110100101**000**1111010011110011010011101101011111000001011011100110100000001110011 10

11101110101100111010111001101**000**01100011100101011111001100100001100100010101 0010

10111**000**0110110001100111011100110110111101111100111010110000110011001010000000110

1010110011101**000**110110111011001001011010010100110111111001100000011111010000001111

1**000**0010011000001100011000100001111001110011110000011001111110011011000100100111

1000101010111000111010110**000**00110000011101010100010110001001101111111001111011001 0

0011101100101110010**000**01100001001111010010011001100001100111010011010000101000111

00111111100110110111011011101010011011011110**00**1111111110101110100110000000100101110

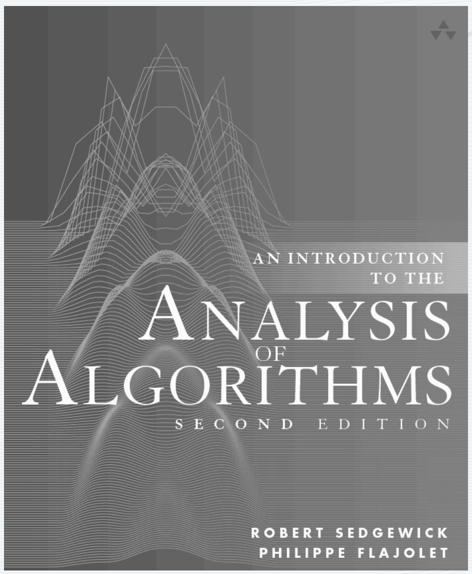
10101**000**1111000010100000110010000011010100101000110011001010101011101101111111110

11**000**00010111101101100010101101011001001000001110111001000000110101000000000101000

1110111101101111101111111111010011101001011111101110100111011**000**11000100100010010

001111111001110101101111100**0**01000100011100001110101111001010111111001110101011111

ANALYTIC COMBINATORICS
PART ONE



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5. Analytic Combinatorics

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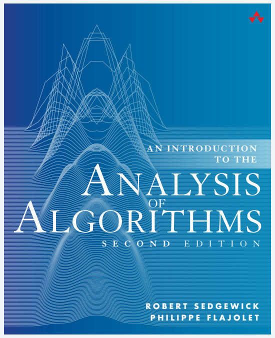
- **Perspective**

Exercise 5.1

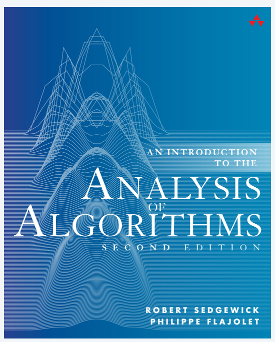
Practice with counting bitstrings.



Exercise 5.1 How many bitstrings of length N have no 000?

Exercise 5.3

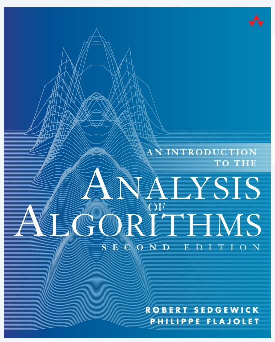
Practice with counting trees.



Exercise 5.3 Let $\mathcal{U}$ be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external), so that the generating function for its counting sequence is $U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \ldots$. Derive an explicit expression for $U(z)$.

Exercise 5.7

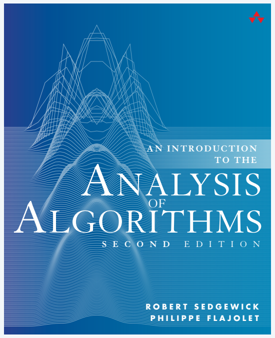
Practice with counting permutations.



Exercise 5.7 Derive an EGF for the number of permutations whose cycles are all of odd length.

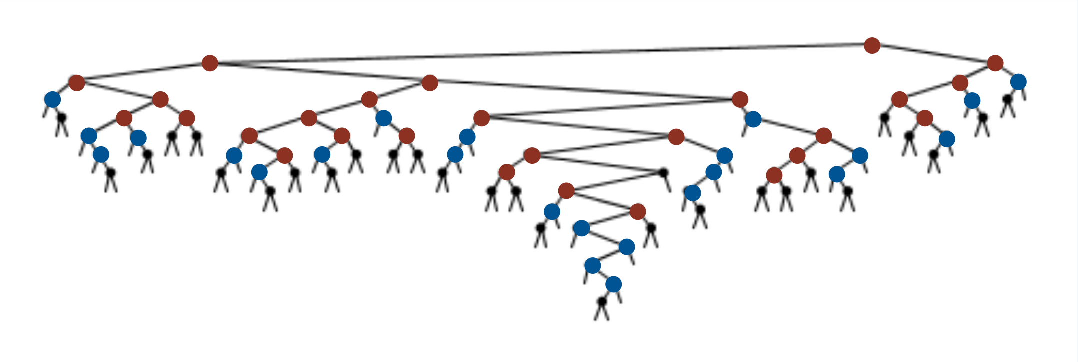
Exercises 5.15 and 5.16

Practice with tree parameters.



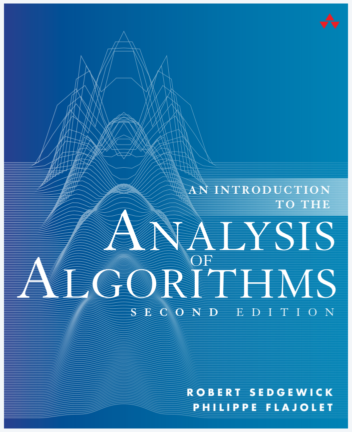
Exercise 5.15 Find the average number of internal nodes in a binary tree of size n with both children internal.

Exercise 5.16 Find the average number of internal nodes in a binary tree of size n with one child internal and one child external.



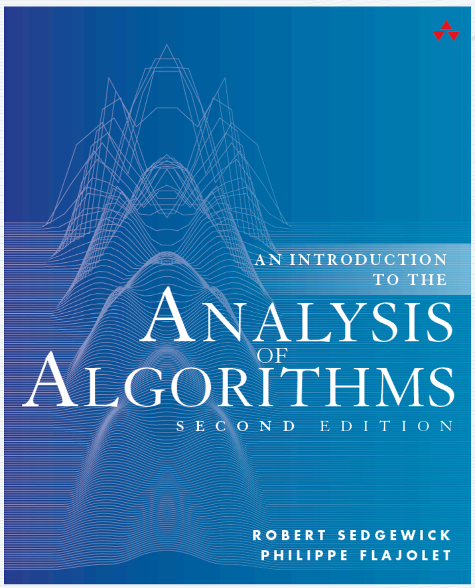
Assignments for next lecture

1. Read pages 219-255 in text.



2. Write up solutions to Exercises 5.1, 5.3, 5.7, 5.15, and 5.16.

ANALYTIC COMBINATORICS
PART ONE



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5. Analytic Combinatorics