AofA Recurrences TEQ 1

Q. Solve the recurrence

$$na_n = (n-3)a_{n-1} + n \quad \text{for} \quad n \geq 3 \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \leq 2$$

n	a_n
3	1
4	5/4
5	3/2
6	7/4

A.

summation factor

$$\overline{n(n-1)(n-2)}a_n = \overline{(n-1)(n-2)}(n-3)a_{n-1} + \overline{n(n-1)(n-2)} \quad \text{for} \quad n \geq 3$$

$$\binom{n}{3}a_n = \binom{n-1}{3}a_{n-1} + \binom{n}{3} \quad \text{for} \quad n \geq 3$$

$$= \sum_{3 \leq k \leq n} \binom{k}{3} \quad = \binom{n+1}{4}$$

$$a_n = \frac{n+1}{4}$$

Note. We try hard to avoid answers that depend on detailed calculations.

AofA Recurrences TEQ 1 (rejected version)

Q. Solve the recurrence

$$na_n = (n-3)a_{n-1} + n \quad \text{for} \quad n \geq \mathbf{4} \quad \text{with} \quad a_n = 0 \quad \text{for} \quad n \leq \mathbf{3}$$

A.

$$n(n-1)(n-2)a_n = (n-1)(n-2)(n-3)a_{n-1} + n(n-1)(n-2) \quad \text{for} \quad n \geq \mathbf{4}$$

$$\binom{n}{3}a_n = \binom{n-1}{3}a_{n-1} + \binom{n}{3} \quad \text{for} \quad n \geq \mathbf{4}$$

$$= \sum_{\mathbf{4} \leq k \leq n} \binom{k}{3} = \binom{n+1}{4} - 1$$

$$a_n = \frac{n+1}{4} - \frac{1}{\binom{n}{3}}$$

n	a_n
4	1
5	7/5
6	17/10
7	

Too complicated for an inclass exam? Probably.

AofA Recurrences TEQ 2

Q. Which of the following is true of the number of compares used by Mergesort ?

$$C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N \quad \text{for} \quad N \geq 2 \quad \text{with} \quad C_1 = 0$$

Order of growth is $N \lg N$	**T**
Exactly $N \lg N$ when N is a power of 2	**T**
Is equal to the number of 1s in the binary representation of the numbers $< N$	**T**
Has periodic behavior	**T**
Is less than $N \lg N + N/4$ for all N	**T**

Some questions are of the form: *Did you watch the lectures and/or do the reading*?

AofA GFs TEQ 1

Q. Match each of the following sequences with their OGF.

0, 0, 1, 3, 6, 10, ...	$\dfrac{1}{1-3z}$
0, 0, 1/2, 0, 1/4, 0, 1/6, ...	$\dfrac{z^2}{(1-z)^3}$
1, 3, 9, 27, 81, 243, ...	$\ln\dfrac{1}{1-z^2}$
1, 1 + 1/2, 1 + 1/2 + 1/3, ...	$\dfrac{3}{1-z}$
3, 3, 3, 3, 3, ...	$\ln\dfrac{1}{1-2z}$
	$\dfrac{1}{1-z}\ln\dfrac{1}{1-z}$
	$\dfrac{1}{(1-z)^3}$

AofA GFs TEQ 2

Q. Suppose that a_n satisfies $a_n = 9a_{n-1} - 20a_{n-2}$ for $n > 1$ with $a_0 = 0$ and $a_1 = 1$

What is $\lim_{n\to\infty} a_n/a_{n+1}$?

A. **5**

$$a(z) = \frac{z}{1 - 9z + 20z^2} = \frac{z}{(1-4z)(1-5z)} = \frac{1}{1-5z} - \frac{1}{1-4z}$$

$$a_n = 5^n - 4^n$$

AofA GFs TEQ 2 (improved form)

Q. Suppose that a_n satisfies $a_n = 9a_{n-1} - 20a_{n-2}$ for $n > 1$ with $a_0 = 0$ and $a_1 = 1$

Fill the box corresponding to the value of $\lim_{n\to\infty} a_n/a_{n+1}$ *and* justify your answer.

○ 3 ○ 4 ● 5

Justification must be 1-3 lines

$$a(z) = \frac{z}{1 - 9z + 20z^2} = \frac{z}{(1-4z)(1-5z)} = \frac{1}{1-5z} - \frac{1}{1-4z}$$

$$a_n = 5^n - 4^n$$

No credit for wrong or unjustified answers.

AofA GFs TEQ 3

Q. Fill the circle corresponding to the value of

$$[z^n] \sum_{0 \le k \le n} \binom{2k}{k} \binom{2n-2k}{n-k}$$

and justify your answer.

○ 2^n ● 4^n ○ $2^{n/2}$

It is $[z^n]\left(\frac{1}{\sqrt{1-4z}}\right)^2$