AofA Strings and Tries Q&A 1

Q. OGF for number of bitstrings not containing 01010 ?

constructions

$$E + (Z_0 + Z_1) \times B = B + P$$

$$Z_{01010} \times B = P + Z_{01} \times P + Z_{0101} \times P$$

GF equations

$$1 + 2zB(z) = B(z) + P(z)$$

$$z^5 B(z) = (1 + z^2 + z^4)P(z)$$

explicit form

$$B(z) = \frac{1 + z^2 + z^4}{z^5 + (1 - 2z)(1 + z^2 + z^4)}$$

AofA Strings and Tries Q&A 2

Q. Rank these patterns by expected wait time in a random bit string.

00000	62
00001	32
01000	34
01010	36
10101	36

$2^P c(1/2)$

AofA Words and Mappings Q&A 1

Q. Find the probability that a random mapping has no singleton cycles.

constructions $C = Z \star SET(C) \qquad M = SET(CYC_{>1}(C))$

EGF equations $C(z) = ze^{C(z)} \qquad M(z) = \exp\Big(\ln \frac{1}{1 - C(z)} - C(z)\Big) \quad = \frac{e^{-C(z)}}{1 - C(z)}$

coefficients via Lagrange inversion

✗ NOT AN EXAM QUESTION (too much calculation)

Lagrange Inversion Theorem (Bürmann form).

If a GF $g(z) = \sum_{n \geq 1} g_n z^n$ satisfies the equation $z = f(g(z))$ with $f(0) = 0$ and $f\,'(0) \neq 0$ then, for any function $H(u)$,

$$[z^n]H(g(z)) = \frac{1}{n}[u^{n-1}]H'(u)\Big(\frac{u}{f(u)}\Big)^n$$

asymptotic result

AofA Words and Mappings Q&A 1 (improved)

Q. Give the EGF for *random mappings with no singleton cycles.*
Express your answer as a function of the Cayley function $C(z) = ze^{C(z)}$

constructions $C = Z \star SET(C) \qquad M = SET(CYC_{>1}(C))$

EGF equations

$$C(z) = ze^{C(z)} \qquad M(z) = \exp\Big(\ln \frac{1}{1-C(z)} - C(z)\Big)$$

$$= \frac{e^{-C(z)}}{1-C(z)}$$

AofA Words and Mappings Q&A 1 (another version)

Q. Find the probability that a *random mapping has no singleton cycles*.
Hint: Do *not* use generating functions.

A. Each entry can have any value but its own index, so the number of *N*-mappings with no singleton cycles is $(N-1)^N$

$$\frac{(N-1)^N}{N^N} = \left(1 - \frac{1}{N}\right)^N$$

$$\sim \frac{1}{e}$$

Related problems (stay tuned)

Q. Find the probability that a *random mapping has no singleton* ***or doubleton*** *cycles*.

	EGF	probability (asymptotic)
all cycle lengths > 1	$M_1(z) = \dfrac{e^{-C(z)}}{1 - C(z)}$	e^{-1}
all cycle lengths > 2	$M_2(z) = \dfrac{e^{-C(z)-C(z)^2/2}}{1 - C(z)}$	$e^{-3/2}$

Rigorous proof requires full mechanism of *singularity analysis* in the complex plane (stay tuned)

Plan for second half of the course

Midterm exam.

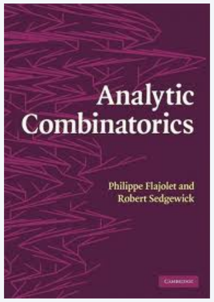
- 3:00-4:20 on Wednesday, March 29, here.
- No surprises (exam is already written).

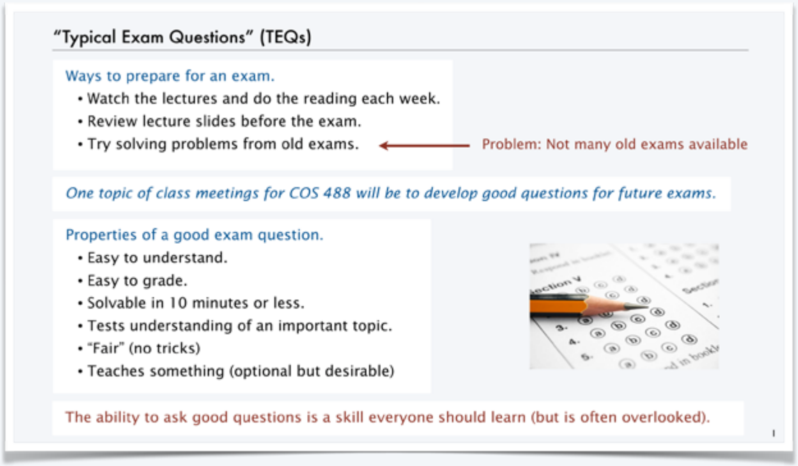
Lectures from Analytic Combinatorics

- One or two lectures, posted weekly.
- Emphasis on lectures, with reference to the book.

New workflow

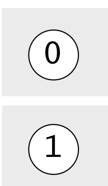
- YOU write a Q&A, due Tuesdays 11:59 PM.
- Smaller problem set due Thursdays 11:59 PM.



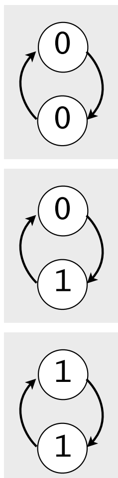


Q&A example: cyclic bitstrings

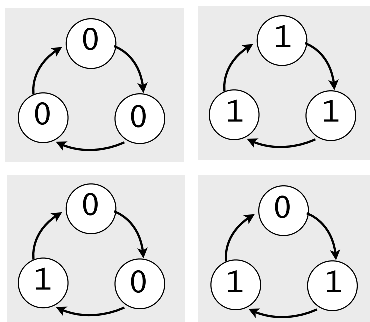
Def. A *cyclic bitstring* is a cycle of bits



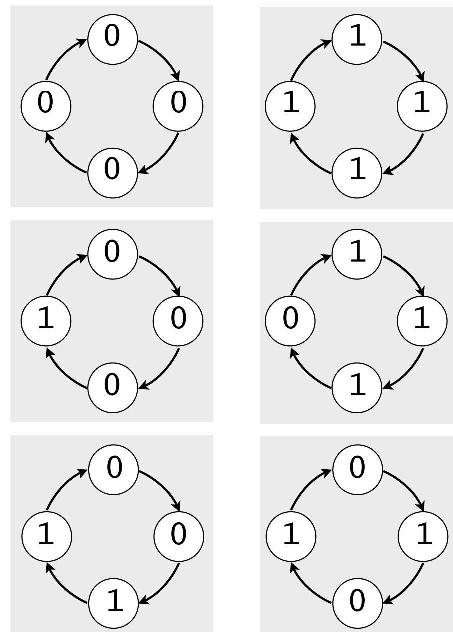
$C_1 = 2$



$C_2 = 3$



$C_3 = 4$



$C_4 = 6$

$C_5 = 8$

Q. How many N-bit cyclic bitstrings ?

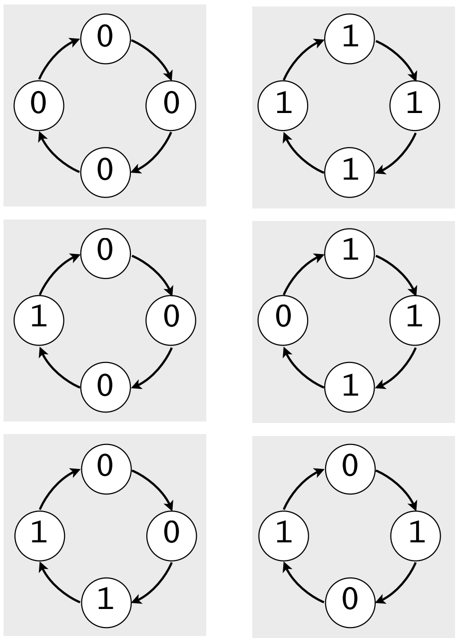
Q&A example: cyclic bitstrings

Q. How many *N*-bit cyclic bitstrings ?

One possibility
- Solution is “easy”.
- Create an exam question with appropriate hints.

Another possibility
- Solution is “difficult” or “complicated”.
- Figure out a way to simplify.
- Or, think about a different problem.

Third possibility
- Problem you thought of is a “classic”.
- Use OEIS.



$C_4 = 6$

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

2, 3, 4, 6, 8

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000031 Number of n-bead necklaces with 2 colors when turning over is not allowed; also number of output sequences from a simple n-stage cycling shift register; also number of binary irreducible polynomials whose degree divides n. +20 83
(Formerly M0564 N0203)

1, **2**, **3**, **4**, **6**, **8**, 14, 20, 36, 60, 108, 188, 352, 632, 1182, 2192, 4116, 7712, 14602, 27596, 52488, 99880, 190746, 364724, 699252, 1342184, 2581428, 4971068, 9587580, 18512792, 35792568, 69273668, 134219796, 260301176, 505294128, 981706832 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Also a(n)-1 is the number of 1's in the truth table for the lexicographically least de Bruijn cycle (Fredricksen).
In music, a(n) is the number of distinct classes of scales and chords in an n-note equal-tempered tuning system. - Paul Cantrell, Dec 28 2011

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LINKS T. D. Noe and Seiichi Manyama, Table of n, a(n) for n = 0..3333 (first 201 terms from T. D. Noe)
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