AofA Analytic Combinatorics Q&A 0

Q. How many binary strings of length N have no occurrence of 01?

A. Trick question!

AofA Analytic Combinatorics Q&A 1

Q. How many binary strings of length N have no occurrence of 011?

A.

Symbolic method: binary strings with restrictions

Ex. How many N-bit binary strings have no two consecutive 0s?

Class	B_{00}, the class of binary strings with no 00
Size	$\lvert b \rvert$, the number of bits in b
OGF	$B_{00}(z) = \sum_{b \in B_{00}} z^{\lvert b \rvert}$

Atoms

type	*class*	*size*	*GF*
0 bit	Z_0	1	z
1 bit	Z_1	1	z

Construction $B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$

"a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00"

OGF equation $B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$

solution $B_{00}(z) = \dfrac{1+z}{1-z-z^2}$

$$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$$ 1, 1, 2, 3, 5, 8, 13, ... ✓

Probably need a 1-page cheatsheet.

$$B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$$

$$B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$$

$$B_{00}(z) = \frac{1+z}{1-z-z^2}$$

B_{011}: class of binary strings with no 011

P_{011}: class of binary strings whose only 011 is at the beginning

EQ 1

$$E + (Z_0 + Z_1) \times B_{011} = B_{011} + P_{011}$$

$$1 + 2zB_{011}(z) = B_{011}(z) + P_{011}(z)$$

Note: true for any pattern p:
prepending a 0 or a 1 to a p-free binary string
gives a p-free binary string
or a binary string whose only occurrence
of p is at the beginning"

EQ 2

$$(Z_0 \times Z_1 \times Z_1) \times B_{011} = P_{011}$$

$$z^3 B_{011}(z) = P_{011}(z)$$

Note: In general, RHS may be more complicated

$$B_{001}(z) = \frac{1}{1 - 2z + z^3} = \frac{1}{(1-z)(1-z-z^2)} = \frac{1}{(1-z)(1-\phi z)(1-\hat{\phi} z)}$$

Probably too intricate for an inclass midterm (borderline, at best)

$$[z^n]B_{001}(z) = \frac{\phi^n}{\sqrt{5}}$$

B_{010}: class of binary strings with no 010

P_{010}: class of binary strings whose only 010 is at the beginning

EQ 1 $$E + (Z_0 + Z_1) \times B_{010} = B_{010} + P_{010}$$

EQ 2 $$(Z_0 \times Z_1 \times Z_0) \times B_{010} = P_{010} + (Z_0 \times Z_1) \times P_{010}$$

if the string from B_{010} begins with 10
constructed string begins with 01010 (two occurrences of 010)
01 followed by a string whose only 010 is at the beginning

AofA Analytic Combinatorics Q&A 1 (improved version)

Q. Match each combinatorial class with a construction.

- binary strings
- binary strings with no 01
- binary strings with no 11
- binary strings with no 001
- binary strings with no 00

$$E + (Z_0 + Z_1) \times B = B + (Z_0 \times Z_1) \times B$$

$$B = E + Z_1 + (Z_0 + Z_1 \times Z_0) \times B$$

$$E + B \times (Z_0 + Z_1) = B + B \times (Z_0 \times Z_0 \times Z_1)$$

$$E + B \times (Z_0 \times Z_1) = B + (Z_0 \times (Z_0 + Z_1)) \times B$$

$$B = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B$$

$$B = E + (Z_0 + Z_1) \times B$$

AofA Analytic Combinatorics Q&A 2 (2015 exam)

Q6. Let $k_n = [z^n]\dfrac{1}{\sqrt{1-3z}}\log\dfrac{1}{1-2z}$. Then:

○ $k_n \sim \log 3\dfrac{3^n}{\sqrt{\pi n}}$

○ $k_n \sim \sqrt{2}\dfrac{2^n}{n}$

with correct version of formula!

$$[z^n]\frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{-n}n^{\alpha-1}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(s+1) = s\Gamma(s)$$

Lesson: Need a 1-page cheatsheet.

AofA Analytic Combinatorics Q&A 2 (improved version)

Q. Match each expression with an approximate value.

$$[z^n]\frac{1}{\sqrt{1-3z}}$$

$$[z^n]\frac{1}{\sqrt{1-3z}}\ln\frac{1}{1-2z}$$

$$[z^n]\frac{1}{(1-2z)(1-3z)}$$

$$[z^n]\ln\frac{1}{\sqrt{1-2z}}$$

$$[z^n]\frac{1}{1-3z}\ln\frac{1}{1-2z}$$

$$\frac{2^{n-1}}{n}$$

$$3^{n+1}$$

$$\frac{3^{n+1}}{n}$$

$$3^n\ln 3$$

$$\frac{3^n}{\sqrt{\pi n}}$$

$$3^n$$

$$\frac{3^n\ln 3}{\sqrt{\pi n}}$$

$$[z^n]\frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)}\rho^{-n}n^{\alpha-1}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(1) = 1$$

$$\Gamma(s+1) = s\Gamma(s)$$

The hard way:

$$[z^n]\frac{1}{(1-2z)(1-3z)} = [z^n]\Big(\sum_{n\geq 0} 2^n z^n\Big)\Big(\sum_{n\geq 0} 3^n z^n\Big)$$

$$= \sum_{0\leq k\leq n} 2^k 3^{n-k}$$

$$= 3^n \sum_{0\leq k\leq n} (2/3)^k$$

Still not convinced?