5. Analytic Combinatorics
Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:
- Analysis begins with formal combinatorial constructions.
- The generating function is the central object of study.
- Transfer theorems can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.

symbolic transfer theorem

combinatorial constructions

generating function equation

analytic transfer theorem

coefficient asymptotics

the “symbolic method”
Analytic combinatorics is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with $N$ nodes?

$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = 1 + zT(z)^2$$

GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient asymptotics
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
The symbolic method
is an approach for translating *combinatorial constructions* to *GF equations*

- Define a *class* of combinatorial objects.
- Define a notion of *size*.
- Define a *GF* whose coefficients count objects of the same size.
- Define *operations* suitable for constructive definitions of objects.
- Develop *translations* from constructions to operations on GFs.

**Formal basis:**
- A *combinatorial class* is a set of objects and a *size function*.
- An *atom* is an object of size 1.
- An *neutral object* is an atom of size 0.
- A *combinatorial construction* uses the union, product, and sequence operations to define a class in terms of atoms and other classes.

**Examples**
- A, B, Z
- | b |
- A(z)
- A × B
- A(z)B(z)

**Building blocks**

<table>
<thead>
<tr>
<th>notation</th>
<th>denotes</th>
<th>contains</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>atomic class</td>
<td>an atom</td>
</tr>
<tr>
<td>E</td>
<td>neutral class</td>
<td>neutral object</td>
</tr>
<tr>
<td>Φ</td>
<td>empty class</td>
<td>nothing</td>
</tr>
</tbody>
</table>
**Unlabelled class example 1: natural numbers**

**Def.** A *natural number* is a set (or a sequence) of atoms.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1 = 1$</td>
<td>$l_2 = 1$</td>
<td>$l_3 = 1$</td>
<td>$l_4 = 1$</td>
<td>$l_5 = 1$</td>
</tr>
</tbody>
</table>

*Unary notation*

---

**Counting sequence**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_N = 1$</td>
<td>$1 \over 1 - z$</td>
</tr>
</tbody>
</table>

**OGF**

\[
\sum_{N \geq 0} z^N = \frac{1}{1 - z}
\]
Def. A bitstring is a sequence of 0 or 1 bits.

Unlabelled class example 2: bitstrings

\[
\begin{array}{c|c}
B_0 &= 1 \\
B_1 &= 2 \\
B_2 &= 4 \\
B_3 &= 8 \\
B_4 &= 16 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
00 & 01 & 10 & 11 \\
00 & 01 & 01 & 11 \\
00 & 01 & 10 & 11 \\
00 & 01 & 10 & 11 \\
\end{array}
\]

Counting sequence

\[
\begin{align*}
B_N &= 2^N \\
\sum_{N \geq 0} 2^N z^N &= \sum_{N \geq 0} (2z)^N = \frac{1}{1 - 2z}
\end{align*}
\]

OGF
**Unlabelled class example 3: binary trees**

**Def.** A binary tree is empty or a sequence of a node and two binary trees.

- \( T_1 = 1 \)
- \( T_2 = 2 \)
- \( T_3 = 5 \)
- \( T_4 = 14 \)

**Counting sequence**

\[
T_N = \frac{1}{N+1} \binom{2N}{N} \quad \frac{1}{2z}(1 - \sqrt{1 - 4z})
\]

**OGF**

Catalan numbers (see Lecture 3)

\[
T(z) = 1 + z T(z)^2
\]
## Combinatorial constructions for unlabelled classes

<table>
<thead>
<tr>
<th>construction</th>
<th>notation</th>
<th>semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>disjoint union</td>
<td>$A + B$</td>
<td>disjoint copies of objects from $A$ and $B$</td>
</tr>
<tr>
<td>Cartesian product</td>
<td>$A \times B$</td>
<td>ordered pairs of copies of objects, one from $A$ and one from $B$</td>
</tr>
<tr>
<td>sequence</td>
<td>$SEQ(A)$</td>
<td>sequences of objects from $A$</td>
</tr>
</tbody>
</table>

Ex 1. \[(00 + 01) \times (101 + 110 + 111) = \begin{array}{cccc} 00101 & 00110 & 00111 & 01101 & 01111 \\
\end{array}\]

Ex 2. \[\bullet \times SEQ(\bullet) = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet 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\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet 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**The symbolic method for unlabelled classes (transfer theorem)**

Theorem. Let $A$ and $B$ be combinatorial classes of unlabelled objects with OGFs $A(z)$ and $B(z)$. Then

<table>
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<tr>
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<td>$SEQ(A)$</td>
<td>sequences of objects from $A$</td>
<td>$\frac{1}{1 - A(z)}$</td>
</tr>
</tbody>
</table>
Proofs of transfers

are immediate from GF counting

\[\sum_{\gamma \in A+B} z^{\mid \gamma \mid} = \sum_{\alpha \in A} z^{\mid \alpha \mid} + \sum_{\beta \in B} z^{\mid \beta \mid} = A(z) + B(z)\]

\[\sum_{\gamma \in A \times B} z^{\mid \gamma \mid} = \sum_{\alpha \in A} \sum_{\beta \in B} z^{\mid \alpha \mid + \mid \beta \mid} = \left( \sum_{\alpha \in A} z^{\mid \alpha \mid} \right) \left( \sum_{\beta \in B} z^{\mid \beta \mid} \right) = A(z)B(z)\]

\[SEQ(A) \equiv \epsilon + A + A^2 + A^3 + A^4 + \ldots \]

\[1 + A(z) + A(z)^2 + A(z)^3 + A(z)^4 + \ldots = \frac{1}{1 - A(z)}\]
**Symbolic method: binary trees**

How many *binary trees* with \( N \) nodes?

<table>
<thead>
<tr>
<th><strong>Class</strong></th>
<th>( T ), the class of all binary trees</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Size</strong></td>
<td>(</td>
</tr>
<tr>
<td><strong>OGF</strong></td>
<td>( T(z) = \sum_{t \in T} z^{</td>
</tr>
</tbody>
</table>

**Atoms**

<table>
<thead>
<tr>
<th>type</th>
<th>class</th>
<th>size</th>
<th>GF</th>
</tr>
</thead>
<tbody>
<tr>
<td>external node</td>
<td>( Z_\Box )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>internal node</td>
<td>( Z_\bullet )</td>
<td>1</td>
<td>( z )</td>
</tr>
</tbody>
</table>

**Construction**

\[
T = Z_\Box + T \times Z_\bullet \times T
\]

**OGF equation**

\[
T(z) = 1 + z T(z)^2
\]

\[
[z^N] T(z) = \frac{1}{N+1} \binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N^3}}
\]

"a binary tree is an external node or an internal node connected to two binary trees"

see Lecture 3 and stay tuned.
Symbolic method: binary trees

How many binary trees with \( N \) external nodes?

<table>
<thead>
<tr>
<th>Class</th>
<th>( T ), the class of all binary trees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>( t ), the number of external nodes in ( t )</td>
</tr>
<tr>
<td>OGF</td>
<td>( T^\square(z) = \sum_{t \in T} z^t )</td>
</tr>
</tbody>
</table>

Atoms

<table>
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<th>type</th>
<th>class</th>
<th>size</th>
<th>GF</th>
</tr>
</thead>
<tbody>
<tr>
<td>external node</td>
<td>( Z^\square )</td>
<td>1</td>
<td>( z )</td>
</tr>
<tr>
<td>internal node</td>
<td>( Z^\bullet )</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Construction

\[ T = Z^\square + T \times Z^\bullet \times T \]

OGF equation

\[ T^\square(z) = z + T^\square(z)^2 \]

\[ T^\square(z) = zT(z) \]

\[ [z^N]T^\square(z) = [z^{N-1}]T(z) = \frac{1}{N} \binom{2N-2}{N-1} \]

“a binary tree is an external node or an internal node connected to two binary trees”

same as \# binary trees with \( N-1 \) internal nodes
Symbolic method: binary strings

Warmup: How many binary strings with \( N \) bits?

<table>
<thead>
<tr>
<th>Class</th>
<th>( B ), the class of all binary strings</th>
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</thead>
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<tr>
<td>Size</td>
<td>(</td>
</tr>
<tr>
<td>OGF</td>
<td>( B(z) = \sum_{b \in B} z^{</td>
</tr>
</tbody>
</table>

**Atoms**

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<thead>
<tr>
<th>type</th>
<th>class</th>
<th>size</th>
<th>GF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 bit</td>
<td>( Z_0 )</td>
<td>1</td>
<td>( z )</td>
</tr>
<tr>
<td>1 bit</td>
<td>( Z_1 )</td>
<td>1</td>
<td>( z )</td>
</tr>
</tbody>
</table>

**Construction**

\[
B = SEQ(Z_0 + Z_1)
\]

**OGF equation**

\[
B(z) = \frac{1}{1 - 2z}
\]

\[
[z^N]B(z) = 2^N \quad \checkmark
\]

"a binary string is a sequence of 0 bits and 1 bits"
Symbolic method: binary strings (alternate)

Warmup: How many binary strings with \( N \) bits?

<table>
<thead>
<tr>
<th>Class</th>
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</tr>
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<tbody>
<tr>
<td>Size</td>
<td>(</td>
</tr>
<tr>
<td>OGF</td>
<td>[ B(z) = \sum_{b \in B} z^{</td>
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**Atoms**

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</tr>
</thead>
<tbody>
<tr>
<td>0 bit</td>
<td>( Z_0 )</td>
<td>1</td>
<td>( z )</td>
</tr>
<tr>
<td>1 bit</td>
<td>( Z_1 )</td>
<td>1</td>
<td>( z )</td>
</tr>
</tbody>
</table>

**Construction**

\[ B = E + (Z_0 + Z_1) \times B \]

**OGF equation**

\[ B(z) = 1 + 2zB(z) \]

**Solution**

\[ B(z) = \frac{1}{1 - 2z} \]

\([z^N]B(z) = 2^N \quad \checkmark \]

“a binary string is empty or a bit followed by a binary string”
Symbolic method: binary strings with restrictions

Ex. How many $N$-bit binary strings have no two consecutive 0s?

Stay tuned for general treatment (Chapter 8)
Symbolic method: binary strings with restrictions

Ex. How many $N$-bit binary strings have no two consecutive 0s?

<table>
<thead>
<tr>
<th>Class</th>
<th>$B_{00}$, the class of binary strings with no 00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>$</td>
</tr>
<tr>
<td>OGF</td>
<td>$B_{00}(z) = \sum_{b \in B_{00}} z^{</td>
</tr>
</tbody>
</table>

Atoms

<table>
<thead>
<tr>
<th>type</th>
<th>class</th>
<th>size</th>
<th>GF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 bit</td>
<td>$Z_0$</td>
<td>1</td>
<td>$z$</td>
</tr>
<tr>
<td>1 bit</td>
<td>$Z_1$</td>
<td>1</td>
<td>$z$</td>
</tr>
</tbody>
</table>

Construction

$B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$

OGF equation

$B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$

solution

$B_{00}(z) = \frac{1 + z}{1 - z - z^2}$

$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$  

“a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00”
Symbolic method: many, many examples to follow

How many ... with ... ?

<table>
<thead>
<tr>
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<th>Atoms</th>
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<tbody>
<tr>
<td></td>
<td>type</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Construction**

- OGF equation
- solution

“a ... is either ... or ... and ...”
5. Analytic Combinatorics

• The symbolic method
• Labelled objects
• Coefficient asymptotics
• Perspective
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
Labelled combinatorial classes

have objects composed of $N$ atoms, labelled with the integers 1 through $N$.

Ex. Different unlabelled objects

Ex. Different labelled objects
Labelled class example 1: urns

Def. An urn is a set of labelled atoms.

\[
\begin{align*}
U_1 &= 1 \\
U_2 &= 1 \\
U_3 &= 1 \\
U_4 &= 1 \\
\end{align*}
\]

\[\sum_{N \geq 0} \frac{z^N}{N!} = e^z\]
Def. A permutation is a sequence of labelled atoms.

Findings:

- $P_1 = 1$
- $P_2 = 1$
- $P_3 = 2$
- $P_4 = 6$

$$\sum_{N \geq 0} \frac{N!z^N}{N!} = \sum_{N \geq 0} z^N = \frac{1}{1-z}$$
**Def.** A *cycle* is a *cyclic sequence* of labelled atoms.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

**Counting sequence**

$$C_N = (N - 1)!$$

**EGF**

$$\sum_{N \geq 1} \frac{(N - 1)! z^N}{N!} = \sum_{N \geq 1} \frac{z^N}{N} = \ln \frac{1}{1 - z}$$
Star product operation

Analog to Cartesian product requires *relabelling in all consistent ways.*

Ex 1. \[
\begin{array}{c}
1 \star 1 2 3 \quad = \quad 1 \ 2 \ 3 \ 4 \quad 2 \ 1 \ 3 \ 4 \quad 3 \ 1 \ 2 \ 4 \quad 4 \ 1 \ 2 \ 3
\end{array}
\]

Ex 2.
## Combinatorial constructions for labelled classes

<table>
<thead>
<tr>
<th>construction</th>
<th>notation</th>
<th>semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>disjoint union</td>
<td>$A + B$</td>
<td>disjoint copies of objects from $A$ and $B$</td>
</tr>
<tr>
<td>labelled product</td>
<td>$A \star B$</td>
<td>ordered pairs of copies of objects, one from $A$ and one from $B$</td>
</tr>
<tr>
<td>sequence</td>
<td>$SEQ(A)$</td>
<td>sequences of objects from $A$</td>
</tr>
<tr>
<td>set</td>
<td>$SET(A)$</td>
<td>sets of objects from $A$</td>
</tr>
<tr>
<td>cycle</td>
<td>$CYC(A)$</td>
<td>cyclic sequences of objects from $A$</td>
</tr>
</tbody>
</table>

A and $B$ are combinatorial classes of labelled objects.
The symbolic method for labelled classes (transfer theorem)

Theorem. Let $A$ and $B$ be combinatorial classes of labelled objects with EGFs $A(z)$ and $B(z)$. Then

<table>
<thead>
<tr>
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<th>EGF</th>
</tr>
</thead>
<tbody>
<tr>
<td>disjoint union</td>
<td>$A + B$</td>
<td>disjoint copies of objects from $A$ and $B$</td>
<td>$A(z) + B(z)$</td>
</tr>
<tr>
<td>labelled product</td>
<td>$A \star B$</td>
<td>ordered pairs of copies of objects, one from $A$ and one from $B$</td>
<td>$A(z)B(z)$</td>
</tr>
<tr>
<td>sequence</td>
<td>$SEQ_k(A)$</td>
<td>$k$-sequences of objects from $A$</td>
<td>$A(z)^k$</td>
</tr>
<tr>
<td></td>
<td>$SEQ(A)$</td>
<td>sequences of objects from $A$</td>
<td>$\frac{1}{1 - A(z)}$</td>
</tr>
<tr>
<td>set</td>
<td>$SET_k(A)$</td>
<td>$k$-sets of objects from $A$</td>
<td>$A(z)^k/k!$</td>
</tr>
<tr>
<td></td>
<td>$SET(A)$</td>
<td>sets of objects from $A$</td>
<td>$e^{A(z)}$</td>
</tr>
<tr>
<td>cycle</td>
<td>$CYC_k(A)$</td>
<td>$k$-cycles of objects from $A$</td>
<td>$A(z)^k/k$</td>
</tr>
<tr>
<td></td>
<td>$CYC(A)$</td>
<td>cycles of objects from $A$</td>
<td>$\ln \frac{1}{1 - A(z)}$</td>
</tr>
</tbody>
</table>
The symbolic method for labelled classes: basic constructions

<table>
<thead>
<tr>
<th>class</th>
<th>construction</th>
<th>EGF</th>
<th>counting sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>urns</td>
<td>( U = \text{SET}(Z) )</td>
<td>( U(z) = e^z )</td>
<td>( U_N = 1 )</td>
</tr>
<tr>
<td>cycles</td>
<td>( C = \text{CYC}(Z) )</td>
<td>( C(z) = \ln \frac{1}{1 - z} )</td>
<td>( C_N = (N - 1)! )</td>
</tr>
<tr>
<td>permutations</td>
<td>( P = \text{SEQ}(Z) )</td>
<td>( P(z) = \frac{1}{1 - z} )</td>
<td>( P_N = N! )</td>
</tr>
<tr>
<td></td>
<td>( P = E + Z \star P )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>construction</th>
<th>notation</th>
<th>EGF</th>
</tr>
</thead>
<tbody>
<tr>
<td>disjoint union</td>
<td>A + B</td>
<td>( A(z) + B(z) )</td>
</tr>
<tr>
<td>labelled product</td>
<td>A \star B</td>
<td>( A(z)B(z) )</td>
</tr>
<tr>
<td>sequence</td>
<td>( \text{SEQ}_k(A) )</td>
<td>( A(z)^k )</td>
</tr>
<tr>
<td></td>
<td>( \text{SEQ}(A) )</td>
<td>( \frac{1}{1 - A(z)} )</td>
</tr>
<tr>
<td>set</td>
<td>( \text{SET}_k(A) )</td>
<td>( A(z)^k/k! )</td>
</tr>
<tr>
<td></td>
<td>( \text{SET}(A) )</td>
<td>( e^{A(z)} )</td>
</tr>
<tr>
<td>cycle</td>
<td>( \text{CYC}_k(A) )</td>
<td>( A(z)^k/k )</td>
</tr>
<tr>
<td></td>
<td>( \text{CYC}(A) )</td>
<td>( \ln \frac{1}{1 - A(z)} )</td>
</tr>
</tbody>
</table>
Proofs of transfers

are immediate from GF counting

\[ A + B \]

\[
\sum_{\gamma \in A + B} \frac{z^{|\gamma|}}{|\gamma|!} = \sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!} + \sum_{\beta \in B} \frac{z^{|\beta|}}{|\beta|!} = A(z) + B(z)
\]

\[ A \star B \]

\[
\sum_{\gamma \in A \times B} \frac{z^{|\gamma|}}{|\gamma|!} = \sum_{\alpha \in A} \sum_{\beta \in B} \left(\frac{|\alpha| + |\beta|}{|\alpha|!}\right) \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!} = \left(\sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!}\right) \left(\sum_{\beta \in B} \frac{z^{|\beta|}}{|\beta|!}\right) = A(z)B(z)
\]

Notation. We write \( A^2 \) for \( A \star A \), \( A^3 \) for \( A \star A \star A \), etc.
Proofs of transfers
are immediate from GF counting

\[ A(z)^k = \sum_{N \geq 0} \#k\text{-sequences of size } N \frac{z^N}{N!} = \sum_{N \geq 0} k \#k\text{-cycles of size } N \frac{z^N}{N!} = \sum_{N \geq 0} k! \#k\text{-sets of size } N \frac{z^N}{N!} \]

<table>
<thead>
<tr>
<th>class</th>
<th>construction</th>
<th>EGF</th>
</tr>
</thead>
<tbody>
<tr>
<td>k-sequence</td>
<td>SEQ_k(A)</td>
<td>( A(z)^k )</td>
</tr>
<tr>
<td>sequence</td>
<td>SEQ_k(A) = SEQ_0(A) + SEQ_1(A) + SEQ_2(A) + \ldots</td>
<td>1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)}</td>
</tr>
<tr>
<td>k-cycle</td>
<td>CYC_k(A)</td>
<td>( \frac{A(z)^k}{k} )</td>
</tr>
<tr>
<td>cycle</td>
<td>CYC_k(A) = CYC_0(A) + CYC_1(A) + CYC_2(A) + \ldots</td>
<td>1 + \frac{A(z)}{1} + \frac{A(z)^2}{2} + \frac{A(z)^3}{3} + \ldots = \ln \frac{1}{1 - A(z)}</td>
</tr>
<tr>
<td>k-set</td>
<td>SET_k(A)</td>
<td>( \frac{A(z)^k}{k!} )</td>
</tr>
<tr>
<td>set</td>
<td>SET_k(A) = SET_0(A) + SET_1(A) + SET_2(A) + \ldots</td>
<td>1 + \frac{A(z)}{1!} + \frac{A(z)^2}{2!} + \frac{A(z)^3}{3!} + \ldots = e^{A(z)}</td>
</tr>
</tbody>
</table>
Labelled class example 4: sets of cycles

Q. How many sets of cycles of labelled atoms?

\[ P^*_1 = 1 \]
\[ P^*_2 = 2 \]
\[ P^*_3 = 6 \]
\[ P^*_4 = 24 \]
Symbolic method: sets of cycles

How many sets of cycles of length \( N \)?

<table>
<thead>
<tr>
<th>Class</th>
<th>( P^* ), the class of all sets of cycles of atoms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>(</td>
</tr>
</tbody>
</table>
| EGF     | \[
P^*(z) = \sum_{p \in P^*} \frac{z^{|p|}}{|p|!} = \sum_{N \geq 0} P^*_N \frac{z^N}{N!}
\]

**Construction**

\[
P^* = SET(CYC(Z))
\]

**OGF equation**

\[
P^*(z) = \exp\left(\ln \frac{1}{1 - z}\right) = \frac{1}{1 - z}
\]

**Counting sequence**

\[
P^*_N = N! [z^N] P^*(z) = N!
\]

<table>
<thead>
<tr>
<th>Atom</th>
<th>type</th>
<th>class</th>
<th>size</th>
<th>GF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>labelled atom</td>
<td>( Z )</td>
<td>1</td>
<td>( z )</td>
</tr>
</tbody>
</table>
Aside: A combinatorial bijection

A permutation is a set of cycles.

Standard representation

Set of cycles representation
Derangements

*N* people go to the opera and leave their hats on a shelf in the cloakroom. When leaving, they each grab a hat at random.

**Q.** *What is the probability that nobody gets their own hat?*

**Definition.** A *derangement* is a permutation with no singleton cycles.
Derangements (various versions)

A group of $N$ people go to the opera and leave their hats in the cloakroom. When leaving, they each grab a hat at random.

**Q.** What is the probability that nobody gets their own hat?

A professor returns exams to $N$ students by passing them out at random.

**Q.** What is the probability that nobody gets their own exam?

A group of $N$ sailors go ashore for revelry that leads to a state of inebriation. When returning, they each end up sleeping in a random cabin.

**Q.** What is the probability that nobody sleeps in their own cabin?

A group of $N$ students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

**Q.** What is the probability that nobody ends up in their own room?
Derangements

are permutations with no singleton cycles.

\[ D_1 = 0 \]
\[ D_2 = 1 \]
\[ D_3 = 2 \]
\[ D_4 = 9 \]
Symbolic method: derangements

How many derangements of length $N$?

<table>
<thead>
<tr>
<th>Class</th>
<th>$D$, the class of all derangements</th>
<th>Atom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>$</td>
<td>p</td>
</tr>
<tr>
<td>EGF</td>
<td>$D(z) = \sum_{d \in D} \frac{z^{</td>
<td>d</td>
</tr>
</tbody>
</table>

Construction   | $D = \text{SET}(\text{CYC}_{>1}(Z))$                                     |

OGF equation    | $D(z) = e^{z^2/2 + z^3/3 + z^4/4 + \ldots} = \exp(\ln \frac{1}{1 - z} - z) = \frac{e^{-z}}{1 - z}$ |

Expansion       | $[z^N] D(z) \equiv \frac{D_N}{N!} = \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} \sim \frac{1}{e}$ |

"Derangements are permutations with no singleton cycles"

"Alternate derivation"  
\[ \text{Set}(Z) \ast D = P \]

Simple convolution

see “Asymptotics” lecture
A group of $N$ students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

**Q.** What is the probability that nobody ends up in their own room?

**A.** $\frac{1}{e} \approx 0.36788$
A group of \( N \) graduating seniors each throw their hats in the air and each catch a random hat.

**Q.** What is the probability that nobody gets their own hat back?

\[
\frac{1}{e} \approx 0.36788
\]
In the hats-in-the-air scenario, a student can get her hat back by "following the cycle".

Q. What is the probability that all cycles are of length $> M$?
Symbolic method: generalized derangements

How many permutations of length $N$ have no cycles of length $\leq M$?

<table>
<thead>
<tr>
<th>Class</th>
<th>$D_M$, the class of all generalized derangements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>$</td>
</tr>
<tr>
<td>EGF</td>
<td>$D_M(z) = \sum_{d \in D_M} \frac{z^{</td>
</tr>
</tbody>
</table>

Atom

<table>
<thead>
<tr>
<th>type</th>
<th>class</th>
<th>size</th>
<th>GF</th>
</tr>
</thead>
<tbody>
<tr>
<td>labelled atom</td>
<td>$Z$</td>
<td>$1$</td>
<td>$z$</td>
</tr>
</tbody>
</table>

Construction

$$D_M = \text{SET}(\text{CYC}_{>M}(Z))$$

OGF equation

$$D_M(z) = e^{\frac{z^{M+1}}{M+1} + \frac{z^{M+2}}{M+2} + \ldots} = \exp(\ln \frac{1}{1 - z} - z - \frac{z^2}{2} - \ldots - \frac{z^M}{M})$$

$$= e^{-z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots - \frac{z^M}{M}}$$

Expansion

$$D_{MN} = \text{?? M-way convolution (stay tuned)}$$
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
Generating coefficient asymptotics

are often *immediately* derived via general "analytic" transfer theorems.

**Example 1. Taylor’s theorem**

**Theorem.** If \( f(z) \) has \( N \) derivatives, then \([ z^N ]f(z) = f^{(N)}(0)/N! \)

**Example 2. Rational functions transfer theorem (see "Asymptotics" lecture)**

**Theorem.** If \( f(z) \) and \( g(z) \) are polynomials, then

\[
[z^n] \frac{f(z)}{g(z)} = -\frac{\beta f(1/\beta)}{g'(\beta)} \beta^n
\]

where \( 1/\beta \) is the largest root of \( g \) (provided that it has multiplicity 1).

**Example 3. Radius-of-convergence transfer theorem**

[see next slide]

Most are based on complex asymptotics. Stay tuned for Part 2
Radius-of-convergence transfer theorem

**Theorem.** If $f(z)$ has radius of convergence greater than 1 with $f(1) \neq 0$, then

$$[z^n] \frac{f(z)}{(1 - z)^\alpha} \sim f(1) \binom{n + \alpha - 1}{n} \sim \frac{f(1)}{\Gamma(\alpha)} n^{\alpha - 1}$$

for any real $\alpha \not\in 0, -1, -2, \ldots$

**Corollary.** If $f(z)$ has radius of convergence greater than $\rho$ with $f(\rho) \neq 0$, then

$$[z^n] \frac{f(z)}{(1 - z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^\alpha n^{\alpha - 1}$$

for any real $\alpha \not\in 0, -1, -2, \ldots$

**Gamma function (generalized factorial)**

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
- $\Gamma(N + 1) = N!$
- $\Gamma(1) = 1$
- $\Gamma(1/2) = \sqrt{\pi}$
Radius-of-convergence transfer theorem: applications

**Corollary.** If \( f(z) \) has radius of convergence \( > \rho \) with \( f(\rho) \neq 0 \), then

\[
[z^n] \frac{f(z)}{(1 - z/\rho)\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^n n^{\alpha - 1}
\]

for any real \( \alpha \not\in 0, -1, -2, ... \)

---

**Ex 1: Catalan**

\[
T(z) = \frac{1}{2z} (1 - \sqrt{1 - 4z})
\]

\[
[z^N] T(z) \sim \frac{4^N}{\sqrt{\pi N^3}}
\]

\( \rho = 1/4 \quad \alpha = -1/2 \quad f(z) = -1/2 \)

\( \Gamma(-1/2) = -2\Gamma(1/2) = -2\sqrt{\pi} \)

**Ex 2: Derangements**

\[
D_M(z) = \frac{e^{-z-z^2/2...-z^M/M}}{1 - z}
\]

\[
[z^N] D_M(z) \sim \frac{N!}{e^{H_M}}
\]

\( \rho = 1 \quad \alpha = 1 \quad f(z) = e^{-z-z^2/2...-z^M/M} \)
Transfer theorems based on complex asymptotics provide universal laws of sweeping generality.

Example: Context-free constructions

A system of combinatorial constructions

\[
\begin{align*}
< G_0 > &= OP_0(< G_0 >, < G_1 >, \ldots, < G_t >) \\
< G_1 > &= OP_1(< G_0 >, < G_1 >, \ldots, < G_t >) \\
& \quad \vdots \\
< G_t > &= OP_t(< G_0 >, < G_1 >, \ldots, < G_t >)
\end{align*}
\]

transfers to a system of GF equations

\[
\begin{align*}
G_0(z) &= F_0(G_0(z), G_1(z), \ldots, G_t(z)) \\
G_1(z) &= F_1(G_0(z), G_1(z), \ldots, G_t(z)) \\
& \quad \vdots \\
G_t(z) &= F_t(G_0(z), G_1(z), \ldots, G_t(z))
\end{align*}
\]

that reduces to a single GF equation

\[
G_0(z) = F(G_0(z), G_1(z), \ldots, G_t(z))
\]

that has an explicit solution

\[
G(z) \sim c - a \sqrt{1 - bz}
\]

that transfers to a simple asymptotic form

\[
G_N \sim \frac{a}{2 \sqrt{\pi N^3}} b^N
\]

Stay tuned for many more (in Part 2).
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
Analytic combinatorics is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with $N$ nodes?

\[
T = E + Z \times T \times T
\]

- **Combinatorial construction**

\[
T(z) = \frac{1}{2z} (1 - \sqrt{1 - 4z})
\]

- **GF**

\[
T_N \sim \frac{4^N}{\sqrt{\pi N^3}}
\]

- **Coefficient asymptotics**
Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with $N$ nodes?

$T = E + Z \times T \times T$

$T(z) = 1 + zT(z)^2$

$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$

**Note:** With complex asymptotics, we can transfer directly from GF equation (no need to solve it). See Part 2.
Old vs. New: Two ways to count binary trees

**Old**

**Recurrence** → **GF**

- **Solving the Catalan recurrence with GFs**
  - Recurrence that holds for all $N$:
    \[ T_N = \sum_{0 \leq k < N} T_k T_{N-k-1} + N \]
  - Multiply by $z^N$ and sum:
    \[ T(z) = \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \left( \sum_{k=0}^{N-1} T_k T_{N-k-1} + N \right) z^N \]
  - Switch order of summation:
    \[ T(z) = 1 + \sum_{N \geq 1} \left( \sum_{k=0}^{N-1} T_k T_{N-k-1} \right) z^N \]
  - Change $N$ to $N-1$:
    \[ T(z) = 1 + \sum_{N \geq 2} \left( \sum_{k=0}^{N-1} T_k T_{N-k-1} \right) z^N \]
  - Distribute:
    \[ T(z) = 1 + z \sum_{N \geq 2} \left( \sum_{k=0}^{N-1} T_k T_{N-k-1} \right) z^{N-1} \]
    \[ T(z) = 1 + z T(z)^2 \]

**New**

- \[ T = E + Z \times T \times T \]
- \[ T(z) = 1 + z T(z)^2 \]
- \[ T_N \sim \frac{4^N}{\sqrt{\pi N^3}} \]
Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many generalized derangements?

\[
D_M = SET(CYC_{>M}(Z))
\]

combinatorial construction

\[
e^{-z-z^2/2-\ldots-z^M/M \over 1 - z}
\]

GF equation

\[
\sim {N! \over e^{H_M}}
\]

coefficient asymptotics
A standard paradigm for analytic combinatorics

Fundamental constructs
- elementary or trivial
- confirm intuition

Compound constructs
- many possibilities
- classical combinatorial objects
- expose underlying structure

Variations
- unlimited possibilities
- *not* easily analyzed otherwise
Combinatorial parameters

are handled as two counting problems via cumulated costs.

Ex: How many leaves in a random binary tree?

1. Count trees

\[ T = E + Z \times T \times T \]

\[ T(z) = \frac{1}{2z} (1 - \sqrt{1 - 4z}) \]

\[ T_N \sim \frac{4^N}{\sqrt{\pi N^3}} \]

2. Count leaves in all trees

\[ T = E + Z \times T \times T \]

\[ T_u(1, z) = \frac{z}{\sqrt{1 - 4z}} \]

\[ C_N \sim \frac{4^{N-1}}{\sqrt{\pi N}} \]

3. Divide

\[ \frac{C_N}{T_N} \sim \frac{N}{4} \]

Symbolic method works for BGFs (see text)
Analytic combinatorics is a calculus for the quantitative study of large combinatorial structures.

Features:
- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.

```
combinatorial constructions  symbolic transfer theorem  generating function equation  analytic transfer theorem  coefficient asymptotics
```

the “symbolic method”
Stay tuned for many applications of analytic combinatorics and applications to the analysis of algorithms.

**Trees**

**Bitstrings**

**Permutations**

**Mappings**
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective
Exercise 5.1

Practice with counting bitstrings.

Exercise 5.1 How many bitstrings of length $N$ have no 000?
Exercise 5.3

Practice with counting trees.

Exercise 5.3 Let $U$ be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external), so that the generating function for its counting sequence is $U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \ldots$. Derive an explicit expression for $U(z)$. 
Exercise 5.7

Practice with counting permutations.

Exercise 5.7 Derive an EGF for the number of permutations whose cycles are all of odd length.
Exercises 5.15 and 5.16

Practice with tree parameters.

**Exercise 5.15** Find the average number of internal nodes in a binary tree of size $n$ with both children internal.

**Exercise 5.16** Find the average number of internal nodes in a binary tree of size $n$ with one child internal and one child external.
Assignments for next lecture

1. Read pages 219-255 in text.

2. Write up solutions to Exercises 5.1, 5.3, 5.7, 5.15, and 5.16.
5. Analytic Combinatorics